The Mathematics Teacher

JANUARY 1961

The logarithm function to the base e—
a development for high school seniors
C. L. SEEBECK, JR.

Construction and evaluation of trigonometric functions of some special angles
JAMES D. BRISTOL

Some challenging series for gifted high school students
ROBERT M. RIPPEY

Does the study of geometry help improve reading ability?
WILLIAM J. FORSETH

Some related theorems on triangles and circles
JOHN D. WISEMAN, JR.

Sep

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.*

Editor and Chairman of the Editorial Board
ROBERT E. PINGRY, *University of Illinois, Urbana, Illinois*

Editorial Board
JACKSON B. ADKINS, *Phillips Exeter Academy, Exeter, New Hampshire*
MILDRED KEIFFER, *Cincinnati Public Schools, Cincinnati, Ohio*
DANIEL B. LLOYD, *District of Columbia Teachers College, Washington 9, D.C.*
E. L. LOFLIN, *Southwestern Louisiana Institute, Lafayette, Louisiana*
ERNEST RANUCCI, *State Teachers College, Union, New Jersey*

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W., Washington 6, D.C.

Officers for 1960–61 and year term expires

President
PHILLIP S. JONES, *University of Michigan, Ann Arbor, Michigan, 1962*

Past-President
HAROLD P. FAWCETT, *Ohio State University, Columbus, Ohio, 1962*

Vice-Presidents
CLIFFORD BELL, *University of California, Los Angeles 24, California, 1961*
MILDRED B. COLE, *Waldo Junior High School, Aurora, Illinois, 1961*
WILLIAM H. GLENN, *Pasadena City Schools, Pasadena, California, 1962*
CLARENCE ETHEL HARDGROVE, *Northern Illinois University, Dekalb, Illinois, 1962*

Executive Secretary
M. H. AHRENDT, *1201 Sixteenth Street, N.W., Washington 6, D.C.*

Board of Directors
FRANK B. ALLEN, *Lyons Township High School, La Grange, Illinois, 1961*
BURTON W. JONES, *University of Colorado, Boulder, Colorado, 1961*
BRUCE E. MESERVE, *Montclair State College, Upper Montclair, New Jersey, 1961*
PHILIP PEAK, *Indiana University, Bloomington, Indiana, 1962*
OSCAR F. SCHAAF, *University of Oregon, Eugene, Oregon, 1962*
HENRY VAN ENGEN, *University of Wisconsin, Madison, Wisconsin, 1962*
J. HOUSTON BANKS, *George Peabody School for Teachers, Nashville, Tennessee, 1963*
IRVIN H. BRUNE, *Iowa State Teachers College, Cedar Falls, Iowa, 1963*
ROBERT E. K. ROURKE, *Kent School, Kent, Connecticut, 1963*

Printed at Menasha, Wisconsin, U.S.A. Second-class postage has been paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 4, 1930. Printed in U.S.A.

UNIVERSITY OF MICH. LIBRARY

FEB 1 1962 O

XUM

ED

The Mathematics Teacher

2284-15

volume LIV, *number* 1 January 1961

The logarithm function to the base e—a development for high school seniors, C. L. SEEBECK, JR. 2

Construction and evaluation of trigonometric functions of some special angles, JAMES D. BRISTOL 4

Some challenging series for gifted high school students, ROBERT M. RIPPEY 8

Does the study of geometry help improve reading ability? WILLIAM J. FORSETH 12

Some related theorems on triangles and circles, JOHN D. WISEMAN, JR. 14

The methods course, GORDON D. MOCK 17

Bright seventh- and eighth-grade children challenge me, ELINOR B. FLAGG 20

The fundamental theorem on linear programming, SHMUEL AVITAL 25

DEPARTMENTS

Experimental programs, EUGENE D. NICHOLS 27
Historically speaking,—HOWARD EVES 32
News notes 38
Points and viewpoints, MARCUS RISKE 40
Reviews and evaluations, KENNETH B. HENDERSON 44
Tips for beginners, JOSEPH N. PAYNE *and* WILLIAM C. LOWRY 47
Have you read? 16; *What's new?* 19
Letter to the editor, 39

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Notice of Annual Business Meeting 26
Report of the Nominating Committee 48
Your professional dates 54

T56

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher*. Institutional subscription to either journal: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N.W., Washington 6, D.C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

XUM

The logarithm function to the base e— a development for high school seniors

C. L. SEEBECK, JR., *University of Alabama, University, Alabama.*

An approach to the logarithm function to the base e *that emphasizes some important properties of the function.*

THE FOLLOWING DEVELOPMENT uses an intuitive concept of area. It has been used successfully with two groups of seniors at a local high school. Previously, the classes had proved that there is a unique function called the increasing logarithm function to the base c, which has the three properties:*

1. $\log_c(ab) = \log_c(a) + \log_c(b)$.
2. There is a number c such that $\log_c(c) = 1$.
3. For all $x > 1$, $\log_c(x) > 0$.

Consider the curve

$$y = \frac{1}{x}.$$

Let A be a function which makes correspond to each positive real number x the measure of the area under the given curve from 1 to x with the convention that if $x > 1$, $A(x) > 0$ and if $x < 1$, $A(x)$ is the negative of the measure of area. We prove that this function has properties (1), (2), and (3) and is therefore a logarithm function.

Property (3) is satisfied immediately by the way in which the function has been defined. For property (2), we note that $A(4)$ is larger than the sum of three rectangles; the first has altitude $\frac{1}{2}$ and has as its base the interval on the x axis from 1 to 2; the second has altitude $\frac{1}{3}$ and base from 2 to 3; the third has altitude $\frac{1}{4}$, and base from 3 to 4. Thus

$$A(4) > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1\ \frac{1}{12}.$$

Since $A(1) = 0$ and since $A(x)$ must take on all values from 0 to at least $1\frac{1}{12}$ there is some number between 1 and 4 where $A(x) = 1$. We call this number e.

To establish property (1) we proceed as follows. Consider a correspondence F between the points of two planes. We let P be a symbol for both a point in the first plane and its coordinates (x, y), and P' for the point and its coordinates (x', y') in the second plane. Then $P' = F(P)$ is some correspondence between the number pairs (x, y) and (x', y'). Let F be defined by the equations

$$x' = \frac{1}{a}x$$

$$y' = ay.$$

Consider a square $ABCD$ in the first plane with vertices (0, 0), (1, 0), (1, 1), and (0, 1). The boundary and interior points of this square are mapped by F into the boundary and interior points of the rectangle $A'B'C'D'$ with vertices

$$(0, 0), \left(\frac{1}{a}, 0\right), \left(\frac{1}{a}, a\right), \text{ and } (0, a).$$

It is clear that the square and the rectangle have the same area, both one. Since

* C. L. Seebeck, Jr., and P. M. Hummel, "Logarithmic and Exponential Functions—a Direct Approach," THE MATHEMATICS TEACHER, LII (October, 1959), pp. 439–443.

the function reduces the scale on the x axis by the factor

$$\frac{1}{a}$$

and magnifies the y scale a times, all unit squares with sides parallel to the axes are mapped into unit rectangles, that is, rectangles with area one. Since areas can be measured by use of rectangular units as well as square units, we make the following assumption.

The function F *maps any closed region of the* xy *plane* R *into a closed region* R′ *of the* x′y′ *plane in such a manner that the area of* R *is equal to the area of* R′.

It may be noted here that the plane $x'y'$ need not be distinct from the plane xy. It is possible to think of the second plane as coinciding with the first plane in such a way that the x' and y' axes coincide with the x and y axes, respectively. Under this interpretation, F maps regions of a plane into other regions of the same plane and it maps curves into other curves.

Now consider the curve

$$y=\frac{1}{x}.$$

The point (1, 1) is mapped into

$$\left(\frac{1}{a}, a\right),$$

a point which lies on the same curve.

$$\left(a, \frac{1}{a}\right)$$

is mapped into (1, 1) and

$$\left(ab, \frac{1}{ab}\right) \text{ into } \left(b, \frac{1}{b}\right),$$

all points on the same curve. Since

$$y=\frac{1}{x}$$

is mapped into

$$\frac{1}{a}y'=\frac{1}{ax'} \text{ or } y'=\frac{1}{x'},$$

every point on the curve is mapped into another point on the curve. In fact, the arc of the curve from

$$\left(a, \frac{1}{a}\right) \text{ to } \left(ab, \frac{1}{ab}\right)$$

is mapped into the arc from (1, 1) to

$$\left(b, \frac{1}{b}\right).$$

Moreover, the region under the first arc is mapped into region under the second arc, and by the assumption about areas these areas are equal. We are now ready to prove that the function $A(x)$ has property (1) of the logarithm function.

By its definition, $A(x)$ is the area under the arc of

$$y=\frac{1}{x} \text{ from (1, 1) to } \left(x, \frac{1}{x}\right).$$

Thus

$A(ab)=A(a)+$area from

$$\left(a, \frac{1}{a}\right) \text{ to } \left(ab, \frac{1}{ab}\right).$$

But this area is equal to $A(b)$ as proved above, and

$$A(ab)=A(a)+A(b).$$

Thus the area function has the three properties which define a logarithm function. The base of the function called e has a value of approximately 2.71828. The graph of the curve near (1, 0) may be drawn by approximating the area function by summing the area of trapezoids taken at intervals of $\frac{1}{10}$.

It is true that most interesting applications of this function and its inverse must await the study of the calculus. However, it is of considerable value for the student to be introduced to this function early in his study, and there is some interest that a function defined in such a manner can have so many of the same properties as the familiar logarithm function to base ten.

Construction and evaluation of trigonometric functions of some special angles

JAMES D. BRISTOL, *Shaker Heights High School, Shaker Heights, Ohio.*

Some interesting ideas relating geometry, trigonometry, and algebra.

THIS EXERCISE in simple high school level mathematics purports to show a means of uniting concepts and processes from various branches of mathematics.

PROBLEM: (*a*) to construct angles of 18°, 36°, 54°, 72°, and 108°; (*b*) to evaluate the trigonometric functions of these angles.

Let AB be a line segment one unit long.

Figure 1

On a working line XY lay off 2 units, $A'B''$.

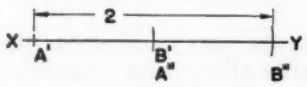
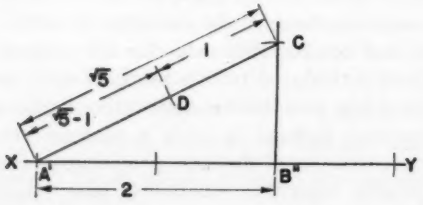


Figure 2

At B'', construct a line perpendicular to XY. On this perpendicular, mark off one unit $B''C$. Join A' to C. By the Pythagorean Theorem, $A'C=\sqrt{5}$ (Fig. 3).

Figure 3



With C as center and compass radius of one unit (from AB again), swing an arc through B'' intersecting $A'C$ at some point D. Since $CD=1$ unit and $A'C=CA'=\sqrt{5}\approx 2.236$ units, D will be a point between A' and C.

By subtraction, $A'D=\sqrt{5}-1$. With A' as center and compass radius of $\sqrt{5}-1$, swing an arc through D and intersecting $A'B''$ at some point E (Fig. 4). $A'E=\sqrt{5}-1\approx 1.236$. Since $A'B''$ is 2 units, E will be a point between A' and B'' on $A'B''$. By subtraction, $EB''=2-(\sqrt{5}-1)=2-\sqrt{5}+1=3-\sqrt{5}$.

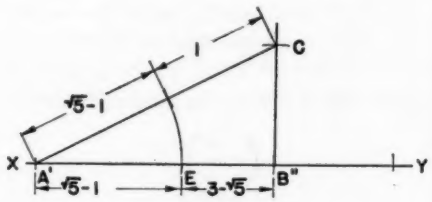


Figure 4

Verify that $A'E$ is the mean proportional between $A'B''$ and EB'', that is,

$$\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}-1}{3-\sqrt{5}}.$$

By rationalizing the denominator on the left and simplifying the left-hand fraction, rationalizing the denominator on the right and simplifying the right-hand frac-

tion, each fraction can be shown to be equal to

$$\frac{\sqrt{5}+1}{2}.$$

We now have four line lengths at our disposal; the original unit length and, from this, 2 units, $\sqrt{5}-1$ units, and $3-\sqrt{5}$ units. A second triangle can now be constructed. On a working line ZW mark off 2 units (where again $AB=1$ unit). Let FG be the 2 units.

Figure 5

With F as center and compass radius at 2 units, swing an arc. With G as center and compass radius at $\sqrt{5}-1$ units (from the first triangle), swing an arc. Call the point of intersection of these two arcs H. Join F to H and G to H with straight lines. See Figure 6.

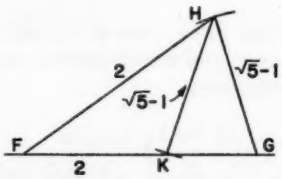


Figure 6

With H as center and compass radius at $\sqrt{5}-1$ units, swing an arc intersecting FG at some point K. This point will lie on FG between F and G. (This can be verified in much the same way as the preceding cases.) Join H to K. There are then two isosceles triangles, FGH and HKG, which are similar, since they share a common base angle G; hence their angles are respectively equal and their sides proportional, thus:

$$\left.\begin{array}{l}\angle F=\angle KHG\\ \angle G=\angle HKG\\ \angle FHG=\angle HGK\end{array}\right\} \text{ and } \frac{FH}{HG}=\frac{HG}{KG};$$

but $FH=2$ and $HG=\sqrt{5}-1$, so

$$\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}-1}{KG}.$$

We can solve this proportion, or we could utilize the proportion obtained from the first triangle and conclude that $KG=3-\sqrt{5}$. By subtraction,

$$FK=2-(3-\sqrt{5})=\sqrt{5}-1.$$

Therefore $FK=KH$ (Fig. 7).

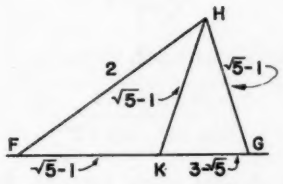


Figure 7

From this, Triangle FKH is isosceles and $\angle F=\angle FHK$. Let $\angle F=\angle FHK=x°$. Then $\angle HKG=2x°$. (An exterior angle of a triangle is equal to the sum of the two remote interior angles.) Since $\angle HKG=\angle G$, $\angle G=2x°$ (Fig. 8). In like

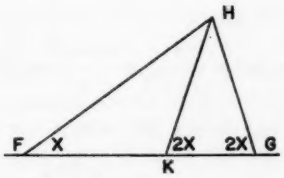


Figure 8

manner, $\angle FHG=2x°$. Since $\angle FHK=x°$, $\angle KHG=x°$.

Substituting in

$$\angle F+\angle G+\angle FHG=180°,$$

we have

$$\begin{aligned} x°+2x°+2x°&=180°\\ 5x°&=180°\\ x°&=36°. \end{aligned}$$

The angles of the triangle are then:

$$\angle F = \angle FHK = \angle KHG = 36°$$
$$\angle FHG = \angle HGK = \angle HKG = 72°$$
$$\angle FKH = 108°.$$

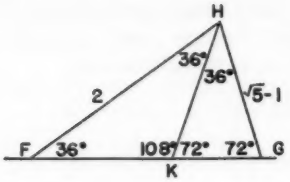


Figure 9

By the law of sines, we have

$$\frac{\sqrt{5}-1}{\sin 36°} = \frac{2}{\sin 72°}.$$

By the double angle formula $\sin 2\theta = 2 \sin \theta \cos \theta$, or $\sin 72° = 2 \sin 36° \cos 36°$,

$$\frac{\sqrt{5}-1}{\sin 36°} = \frac{2}{2 \sin 36° \cos 36°}$$

which simplifies to

$$\frac{\sqrt{5}-1}{1} = \frac{1}{\cos 36°}.$$

Thus

$$\cos 36° = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4} \approx .809.$$

Since $\cos \theta = \sin (90° - \theta)$,

$$\cos 36° = \sin 54° = \frac{\sqrt{5}+1}{4}.$$

From $\sin^2 \theta + \cos^2 \theta = 1$, we can now compute sin 36° or cos 54° and, by division, tan 36° and tan 54°. By the half-angle formulas, namely

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1-\cos \theta}{2}}$$
$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1+\cos \theta}{2}},$$

sin 18°, cos 18°, and tan 18° (and, of course, the reciprocal functions) can be computed.

A second procedure follows: From H, construct a line perpendicular to FG. Call this line segment HM. HM will bisect KG, making

$$MG = \frac{3-\sqrt{5}}{2}$$

(Fig. 10). By the Pythagorean Theorem,

$$HM = \sqrt{(\sqrt{5}-1)^2 - \frac{(3-\sqrt{5})^2}{4}}.$$

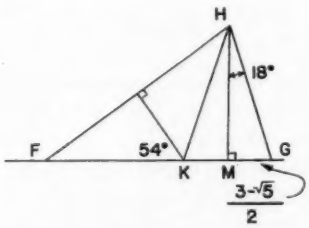


Figure 10

With the three sides of triangle MHG now known, the trigonometric functions can be computed.

$$\sin 18° = \frac{\left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}-1} = \frac{3-\sqrt{5}}{2(\sqrt{5}-1)} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1}$$
$$= \frac{3\sqrt{5}+3-5-\sqrt{5}}{2(5-1)} = \frac{2\sqrt{5}-2}{2(4)}$$
$$= \frac{\sqrt{5}-1}{4}$$

$$\sin 18° = \cos 72° = \frac{\sqrt{5}-1}{4}$$

cos 18° or sin 72° can be computed in a similar manner. The other four functions follow.

One interesting trigonometric identity which now can be easily established is:

$$\sin 54° - \sin 18° \equiv \tfrac{1}{2}.$$

Substituting, we have

$$\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}=\frac{\sqrt{5}+1-\sqrt{5}+1}{4}$$

$$=\frac{2}{4} \text{ or } \frac{1}{2}.$$

Another interesting observation is to compare the roots of $4x^2+2x-1=0$ with some of the trigonometric ratios found above. The roots are

$$\frac{-1+\sqrt{5}}{4} \text{ and } \frac{-1-\sqrt{5}}{4}.$$

Note that the first is the sin 18° and that the second is the negative of the sin 54°. In terms of roots, then, the equation can be written:

$$(x-\sin 18°)(x+\sin 54°)=0$$

or

$$x^2+(\sin 54°-\sin 18°)x$$
$$-(\sin 18°)(\sin 54°)=0.$$

Since it can be shown that (sin 18°) (sin 54°) $=\frac{1}{4}$, we have

$$x^2+\tfrac{1}{2}x-\tfrac{1}{4}=0$$

or

$$4x^2+2x-1=0.$$

There is another by-product of these constructions, namely, the construction of three regular polygons.

1. A regular inscribed pentagon. A central angle formed by radii to two consecutive vertices is 72°. Each interior angle is 108°. Both of these angles have been constructed above.

2. A regular inscribed decagon. A central angle formed by radii to two consecutive vertices is 36°. Each interior angle is 144°. We have the first of these. The second is the supplement of the first and so can be obtained.

3. A regular inscribed pentadecagon. A central angle is 24°. This, however, can be written as 60°−36°. We can construct both of these angles—the former from an equilateral triangle—so we can construct a 24° angle, and from there a regular pentadecagon.

The following method for the construction of the regular pentagon is commonly outlined, but seldom proved.

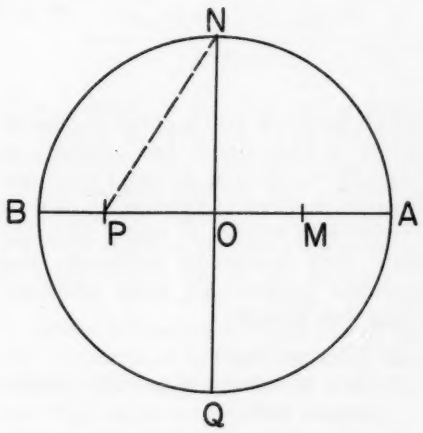


Figure 11

Construct a circle O. (For us, let the radius be 2 units.) Construct two diameters perpendicular to each other, BA and NQ (Fig. 11). Construct the midpoint M of OA. With M as center and MN as radius, swing an arc intersecting BA at some point P. PN is the desired line length for a side of the regular inscribed pentagon.

The proof follows easily from the preceding discussion and might be a good exercise for the student.

We have constructed angles of 18°, 36°, 54°, 72°, and 108°. By addition, we can construct any other angles of the form $18n°$ for n a positive integer. We have evaluated, in order, cos 36°, sin 54°, sin 18°, cos 72°. From these and with the use of the half-angle, double-angle, sum, and difference formulas we can evaluate the trigonometric functions of any angle $18n°$, n a positive integer.

Some challenging series for gifted high school students

ROBERT M. RIPPEY, *Lyons Township High School, LaGrange, Illinois.*

Opportunities for student discovery and creation are abundant in a study of series as described in this article.

WHILE WORKING with a group of selected juniors of high ability and ambition at Lyons Township High School, I found the ideas which follow of value in stimulating interest with respect to series. After the usual work concerning arithmetic and geometric progressions, three additional topics were studied:

A. Fibonacci series
B. Sum of squares of positive integers from 1 to n
C. A generalization of figurate numbers

FIBONACCI SERIES

Fibonacci series are constructed by specifying the first two terms and then making the third term equal to the sum of the first and second, the fourth term equal to the sum of the second and third, and so on. For example, the Fibonacci series with a first term of 5 and a second term of 3 would be

$$5+3+8+11+19+\cdots \quad (1)$$

A particular example which students may have seen before in geometry in connection with the law of phyllotaxy is:

$$1+1+2+3+5+8+13+\cdots \quad (2)$$

It can be shown by a good student that any two consecutive terms of this series are relatively prime. This affords an exercise of some challenge. In examining the structure of the series, students can observe the following relationship. Let the first term be A and the second term B. Then any Fibonacci series becomes

$$(A)+(B)+(A+B)+(A+2B)+(2A+3B)+(3A+5B)\cdots. \quad (3)$$

If this series is compared with (2) it can be seen that the ith term of the general series

$$T_i=AT_{i-2}+BT_{i-1}. \quad (4)$$

where T_{i-2} and T_{i-1} are terms from series (2).

The formula for the terms in series (2) can form a topic for considerable speculation. Few students if any would hit upon the formula for the ith term, which is:

$$\frac{(1+\sqrt{5})^i-(1-\sqrt{5})^i}{2^i\sqrt{5}}.$$

To verify this, the student can show that T_1 and T_2 equal 1 and also that $T_i=T_{i-1}+T_{i-2}$ for all i greater than or equal to 3. Applying this in (4), the student discovers that the equation for the ith term on any Fibonacci series is:

$$T_i=\frac{(1+\sqrt{5})^{i-2}-(1-\sqrt{5})^{i-2}}{2^{i-2}\sqrt{5}}\cdot A+\frac{(1+\sqrt{5})^{i-1}-(1-\sqrt{5})^{i-1}}{2^{i-1}\sqrt{5}}\cdot B.$$

If the instructor is inclined toward a more spectacular presentation, he can display his calculating powers by challenging a student to an addition contest. A student volunteer is given two numbers at random from the class. He is asked to construct ten terms of the series, using the

given numbers as T_1 and T_2, respectively. He is then told to add the ten terms in competition with the instructor who will perform the calculation mentally.

Of course, the instructor can write down the sum as soon as the seventh term of the series is written down for the sum of ten terms is always eleven times the seventh term. Proof of this can be left to the class as soon as they are convinced that trickery is involved.

The sum of i terms of the series is somewhat more complicated to develop. However, if a student prepares the accompanying table, he may observe the pattern involved.

TABLE 1

i	Terms	Sum of i terms
1	A	
2	B	$A+B$
3	$A+B$	$2A+2B$
4	$A+2B$	$3A+4B$
5	$2A+3B$	$5A+7B$
6	$3A+5B$	$8A+12B$
7	$5A+8B$	$13A+20B$

Most students will recognize the coefficients of the A's in the sum column. However, the coefficients of the B's are more puzzling. If necessary, a hint is in order. If the successive differences of the B coefficients are taken, they form the sequence 1, 2, 3, 5, 8. . . . Further study of the sum indicates that the ith coefficient of B is the sum of the first i terms of series (2). What then is the sum of n terms of series (2)?

$$\sum_{i=1}^{n} T_i = T_n + T_{n+1} - 1$$

The student should convince himself of this fact by working with specific examples.

While it is true that

$$\sum_{1}^{n} T_i = T_n + T_{n+1} - 1$$

for the series where $A=1$ and $B=1$, it is not true for all Fibonacci series, as can easily be seen using series (1) as a counterexample. The sum of $n+1$ terms of any Fibonacci series having A for the first term and B for the second becomes, by substitution:

$$\sum_{1}^{n+1} T_i = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} A + \left[\frac{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}{2^{n}\sqrt{5}} + \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} - 1 \right] B.$$

This formula, so far, has had sufficient complexity to satisfy even the most critical students. For students desiring to extend themselves further, it is interesting to consider the possibility of simpler formulas for this sum if certain relationships exist between A and B: for example, if $A=B$, or if $A=B+1$. Also, an approximate computation of the sum of the first 271 terms of the series starting with 128 and 345, using logarithms, is a problem which appeals to certain students.

Sum of squares of positive integers

The problems encountered in analyzing and deriving the formulas for the sums of n terms of the squares of the positive integers can lead a student into many interesting byways. The basic approach to this subject is to analyze first the series $1^0+2^0+3^0+\cdots+n^0$ and recognize that the nth term is 1 and the sum of n terms is n. Knowing that $1^0+2^0+3^0+\cdots+n^0=n^1$, we may write the equations:

$$\begin{aligned}
n^2-(n-1)^2 &= 2n-1 \\
(n-1)^2-(n-2)^2 &= 2(n-1)-1 \\
(n-2)^2-(n-3)^2 &= 2(n-2)-1 \\
&\vdots \\
2^2-1^2 &= 2(2)-1 \\
1^2-0^2 &= 2(1)-1 \\
\hline
n^2 &= 2\left(\sum_{x=1}^{n} x\right) - n
\end{aligned}$$

$$\frac{n^2+n}{2} = \sum_{x=1}^{n} x.$$

A geometrical interpretation of this formula can be given as follows: The numbers from 1 to n can be pictured as a staircase with risers of one unit. Combining two sets of staircases gives us a rectangle $n(n+1)$ in area (Fig. 1). Since the area of the rectangle represents the sum of two equal series,

$$\sum_{x=1}^{n} x=\frac{n(n+1)}{2}.$$

Since we now know the formula for the sum of the positive integers from 1 to n, we can derive the sum of the squares of the positive integers from 1 to n in the same way.

$$n^3-(n-1)^3=3n^2-3n+1$$
$$(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1$$
$$(n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$2^3-1^3 = 3(2)^2-3(2)+1$$
$$1^3-0^3 = 3(1)^2-3(1)+1$$

$$n^3=3\sum_{x=1}^{n} x^2-3\sum_{x=1}^{n} x+n$$

$$n^3+3\frac{(n^2+n)}{2}-n=3\sum_{x=1}^{n} x^2$$

$$\frac{2n^3+3n^2+n}{6}=\sum_{x=1}^{n} x^2.$$

In like manner the students show

$$\sum_{x=1}^{n} x^3=\tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}n^2$$

and

$$\sum_{x=1}^{n} x^4=\tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\frac{1}{30}n.$$

The students are given some of the formulas for higher powers for reference. At this point it may be desired to make some conjectures regarding the structure of the formulas. Also, some students may want to find the formula for the sum of n terms of a series whose nth term can be expressed as a polynomial.

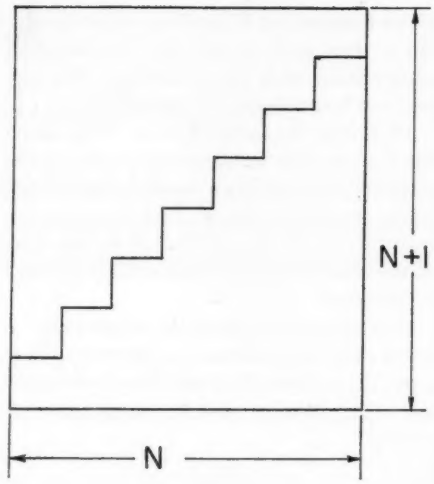


Figure 1

Series derived from figurate numbers

I like to introduce my students to this type of series by asking the question, "How many cannonballs can be stacked in a pile which is 15 cannonballs high?" I find this type of problem particularly appealing, for it is solved more easily if a general solution is considered, than if brute force methods are applied.

There is another advantage. This is also a time to repeat the desirability of simplification in looking for the solution of a problem.

The student should first ask himself or be asked, "Can I solve a simpler problem which is much like this one?" This simplification can be obtained by considering a smaller number of cannonballs and a smaller number of dimensions. If we work with one dimension, the number of

cannonballs in the nth position is 1. The total number of cannonballs up to the nth one is n (Fig. 2). Now if the cannonballs are placed in a triangular pattern on the ground, the number in each row corresponds to n, the number of the row, and the sum of the number of cannonballs up to the nth row is equal to the sum of the power series, $1+2+3+\cdots+n$, or

$$\frac{n(n+1)}{2}.$$

The triangular pattern is illustrated in Figure 3.

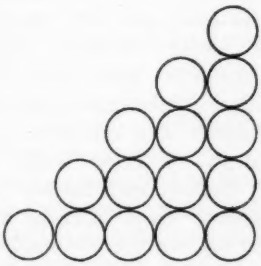
The number in the ith position is 1:

$${}_1T_i=1.$$

The total number up to and including the nth position is:

$$\sum_{i=1}^{n} {}_1T_i=n.$$

Figure 2



The number in the ith row is i:

$${}_2T_i=i.$$

The total of n rows is:

$$\sum_{i=1}^{n} {}_2T_i=\frac{n(n+1)}{2}.$$

Figure 3

At this point, it is profitable to consider the arrangements of the cannonballs in square, pentagonal, and other configurations on the ground.

One cannonball stack with equilateral triangle base turns out to have 1 cannonball in the top plane, 3 in the second, 6 in the third, and 10 in the fourth plane. Therefore,

$${}_3T_i=\frac{n(n+1)}{2},$$

and

$$\sum_{i=1}^{n} {}_3T_i=\sum_{i=1}^{n}\left(\frac{i^2}{2}+\frac{i}{2}\right)$$

$$=\frac{2n^3+3n^2+n}{12}+\frac{n^2+n}{4}=\frac{n(n+1)(n+2)}{6}.$$

The solution to our problem concerning a pile of cannonballs 15 balls high then is, for this particular pile,

$$\frac{(15)(16)(17)}{6}.$$

In fact, for any number of layers of cannonballs arranged as a regular tetrahedron, the total number of cannonballs may now be computed. For students desirous of doing additional study, reference can be made at this point to the mathematical analysis of crystal structures. The treatment of the problem for arrangements of the elements in cubical and other configurations is also possible.

The series which have been discussed will certainly not appeal to all gifted students. One characteristic of the gifted student is his greater selectivity toward ideas. To exploit this characteristic, these topics may prove of value in working with some students. For, although their rate of rejection of ideas may at times be high, an idea which will motivate and produce deep interest may force the gifted student to master a set of more general objectives in order to satisfy a particular urge for understanding a problem not in the main stream of his syllabus.

Does the study of geometry help improve reading ability?

WILLIAM J. FORSETH, *Bemidji State College, Bemidji, Minnesota.*

Is there any special characteristic of geometry study that helps students become better readers than they would be without geometry study?

CERTAIN ASPECTS of solving a geometry problem are like reading a paragraph in print. Thorndike associated mathematics and reading when he said:

> *Understanding a paragraph is like solving a problem in mathematics* [italics not in the original]. It consists in selecting the right elements of the situation and putting them together in the right relations, and also with the right weight or influence or force for each. The mind is assailed as it were by every word in the paragraph. It must select, repress, soften, emphasize, correlate, and organize, all under the influence of the right mental set or purpose or demand.[1]

Opinions such as this suggested that there might be a relationship between the study of geometry and reading ability. The problem was this: Do students who study geometry show more improvement in reading ability than students who do not study geometry? Teachers who were asked for their opinions did not agree, and their comments indicated strong bias and prejudice. Mathematics teachers generally held that geometry would help improve reading ability, whereas English teachers and college education professors generally held the opposite view. These opinions could not be used to draw conclusions except to show the need for an objective study of the problem.

An attempt was made to find an answer through an analysis of reading test scores. Tenth-grade students from three Minnesota high schools were tested in the fall and again the following spring, and their scores were compared. The statistical technique used was the method of equivalent groups, matched by pairs. Equivalent groups, matched student by student for intelligence, initial reading ability, and school, were compared for amount of reading improvement made during the year. The matching was carefully done so that the two groups were almost identical in intelligence and initial reading, thereby controlling these factors as contributing influences.

Comparisons between equivalent groups were made on the Schrammel-Gray High School and College Reading Test for each of three test scores: reading gross comprehension, reading rate, and reading comprehension-efficiency. Geometry students were compared with non-geometry students, biology students with non-biology students, home economics students with non-home economics students, and industrial arts students with non-industrial arts students. Similar comparisons for English and history were not made because these subjects were constants for all tenth-grade students. The fact that English and history were constant subjects made them common factors and thereby ruled them out as contributing influences in the comparisons.

The gain made by a group over its equivalent group—the difference between means of end test scores—was tested for significance. In the three equivalent-

[1] E. L. Thorndike, "Reading as Reasoning," *Journal of Educational Psychology*, VIII (June 1917), 332.

TABLE 1

SUMMARY OF READING GAIN AND *t* RATIOS OF DIFFERENT GROUPS

Group	*Difference between means*	*t ratios observed*	*t ratio needed for significance (.05 level)*	*No. of pairs*
Geometry				
Gross comprehension	4.38	3.06	1.99	79
Rate	8.41	2.49	1.99	80
Comprehension-efficiency	2.55	2.52	1.99	82
Biology				
Gross comprehension	1.49	1.00	2.00	64
Rate	2.49	.66	2.01	55
Comprehension-efficiency	1.05	.91	2.00	71
Home Economics				
Gross comprehension	−2.48	1.21	2.04	29
Rate	−9.83	1.85	2.04	30
Comprehension-efficiency	.52	.24	2.04	29
Industrial Arts				
Gross comprehension	2.32	.89	2.08	22
Rate	−1.57	.21	2.10	19
Comprehension-efficiency	− .04	.02	2.07	23

group comparisons between geometry and non-geometry groups, the geometry groups showed gains over the non-geometry groups which were significant. The difference between means in reading gross comprehension gave a *t* ratio of 3.06, which was greater than the *t* at the .01 level, and the difference between means for reading rate and reading comprehension-efficiency showed *t* ratios of 2.49 and 2.52, which were greater than the *t* needed for significance at the .05 level. In the nine group comparisons made for biology, home economics and industrial arts, none of the gains in reading was significant, nor did any approach significance. Table 1 summarizes reading gains and *t* ratios made by the different groups.

This statistical study had limitations in sampling and in controlling all the factors which may have influenced results. Students of only three geometry teachers were involved in the study. These teachers may have been exceptionally good teachers who did a better than average job of inspiring students in all phases of school performance, including improvement in reading. Matching students for intelligence and initial reading ability did not remove entirely the influence of the bright student who could not be matched in the other group and was therefore excluded, because this bright student may have been an important stimulus to the learning of classmates. The reverse could be said for the dull student who could not be matched; he may have slowed down the progress of his group. These were the main limitations of this study. There were others, but these were the most apparent and probably the most important of the limitations involved.

Within the limitations of this study, this conclusion appeared to have some support: the pupils who study geometry improve in reading ability slightly more than do their classmates of equal initial reading ability and intelligence who study subjects other than geometry. This conclusion did not imply that the study of geometry can substitute as a remedial reading technique, nor did it mean that a student should elect geometry to improve his reading. The small gain in reading ability made by geometry students, even though statistically significant, indicated that a more direct approach to reading improvement would be more profitable.

Some related theorems on triangles and circles

JOHN D. WISEMAN, JR., *Tenafly High School, Tenafly, New Jersey.*

The words are different but the theorems are about the same configurations of lines and points.

HIGH SCHOOL GEOMETRY is concerned with patterns, or the relationships between various figures. Teaching high school students to recognize these patterns is an important part of the geometry course. This article is concerned with certain theorems about isosceles triangles. The basic pattern of these theorems reappears in other subject matter areas of geometry, notably as circle theorems.

In the usual course in high school geometry, the theorems on congruent triangles are followed by the theorem which states that the base angles of an isosceles triangle are equal (or congruent). The proof of this theorem may use Figure 1.

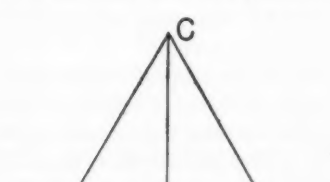


Figure 1

If $AC \cong BC$ and if CD is a median, then it can be proved that CD is an altitude and that CD bisects $\angle C$. There are six such theorems in which it is assumed that CD is either a median, an altitude, or an angle bisector, and in which it is proved that CD is also the other two.

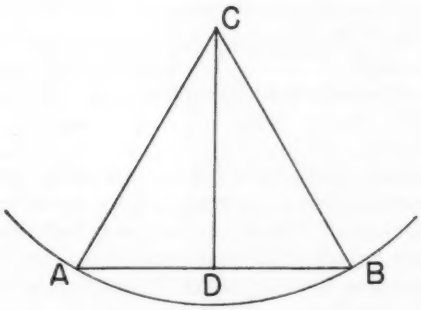


Figure 2

The statements of these theorems use words which are defined in terms of a triangle. For example:

If in an isosceles triangle there is a median to the base, then the median is also an altitude to the base.

Such words as "isosceles," "median," and "base" are defined in relation to the sides and angles of a triangle.

Now by making a simple change in the figure it is possible to transform the previous theorem into a theorem about a circle. If point C is the center of a circle and the circle is on point A and point B, then the figure can be described by words which are defined in terms of a circle (Fig. 2). The common vertex of the isosceles sides is now the center of the circle; the equal (or congruent) sides are radii; the

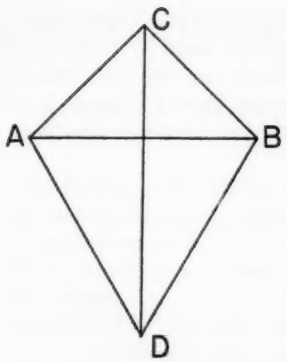


Figure 3

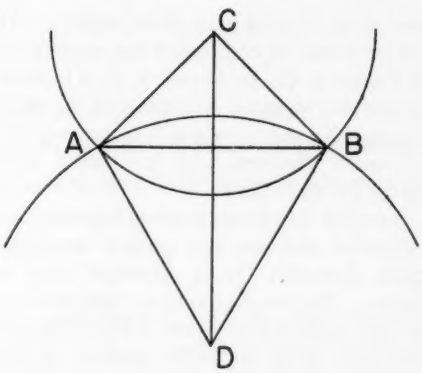


Figure 4

base is a chord; and so on. Each word or phrase in the isosceles-triangle theorem has a counterpart in the statement of the circle theorem. This means that for each triangle theorem there is a corresponding circle theorem. The basic pattern is the same in each set of theorems; the difference is only in the language used to state the theorems. Thus the triangle theorem stated previously becomes the following circle theorem:

If there are two radii of a circle with a connecting chord and if a third radius bisects the chord, then the third radius is perpendicular to the chord.

The proofs of this circle theorem and its corresponding triangle theorem are basically the same. In both theorems two triangles are proved congruent in the same way. The differences are mainly the words used to name the parts of the figure. This knowledge should make it easier for students to prove the circle theorem.

These isosceles-triangle theorems are not the only theorems that correspond to certain circle theorems. Another example is the two-point theorem or the perpendicular-bisector theorem (Fig. 3):

If two points are equidistant from the ends of a straight line segment, then the two points are on the perpendicular bisector of the line segment.

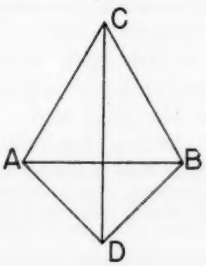


Figure 5

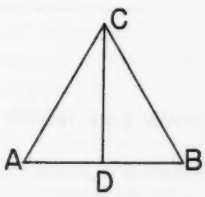


Figure 6

This theorem corresponds to the following circle theorem (Fig. 4):

If two circles intersect and there is a common chord, then their line of centers is the perpendicular bisector of the common chord.

Again the basic pattern is the same for each theorem; they differ only in language. In fact, the easiest way to prove the circle theorem is to establish first that point C and point D are equidistant from the two points of intersection of the circle. Then the two-point theorem can be used immediately to complete the proof.

There is also a relationship between the two-point theorem and the isosceles-triangle theorems, the latter being a special case of the two-point theorem. Figures 5 and 6 may be used in the proof. There are

two ways to compare these figures. (1) Figure 6 may be considered the upper part of Figure 5. (2) In Figure 6, D is located on the line segment AB, while in Figure 5 it is not. No matter where D is located, the two-point theorem can be stated correctly for each figure.

Another relationship exists between the two-point theorem and certain parallelogram theorems. In a rhombus (and a square), the two diagonals are perpendicular bisectors of each other. These diagonal theorems have the same pattern as the two-point theorem. They differ only in the language used to state the theorem. It is evident that the basic pattern of this one theorem alone reappears in a number of different theorems in Euclidean geometry.

This article has considered certain theorems in one subject matter of geometry which have their counterparts in other subject matter of geometry. A basic pattern such as this is a useful tool in the discovery and proof of various theorems in different areas of geometry.

Have you read?

BUSEMANN, HERBERT. "The Role of Geometry for the Mathematics Student," *The American Mathematical Monthly*, March 1960, pp. 281–285.

The trend seems to be in the direction of less and less geometry. Mr. Busemann's article presents the case for geometry very well, and I feel all secondary-mathematics teachers should read it. His study shows that of 182 colleges, 13 offer no geometry at all, 58 offer analytic geometry, 6 offer differential, 29 offer non-Euclidean, and 52 offer geometry of the triangle and circle. The majority of schools offer two or less courses in geometry.

What about the factual information now in geometry? Does each new approach demand that the old be junked? Will present mathematics be junked in a few years? Will abstract mathematics be possible if we fail to take advantage of geometric intuition? And what about the necessary training in continuity?

Read this article—if you have planned to delete geometry from your program, you may give it a second thought.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

"How They Size You Up for Credit," *Changing Times, The Kiplinger Magazine*, July 1960, pp. 29–31.

In the teaching of mathematics we often find ourselves involved in mathematics as it relates to the everyday business. This article is excellent background material as you go into consumer buying or installment credit. What does it take to be granted credit? Why are complete names and correct spellings necessary, what makes age important, or where you live? You can see how income is important, but what per cent of your income can you borrow, what per cent can you obligate per month on debts, and does it help to have had credit previously? What type of occupation ranks highest with the credit manager? If you want to know how to stay in the good graces of the credit manager, then read this article. It will help your students work their mathematics and it will help you keep a good credit rating.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

KEMP, C. GRATTON. "Effect of Dogmatism on Critical Thinking," *School Science and Mathematics*, April 1960, pp. 314–319.

I believe we all subscribe to a need for critical thinking, but do we all define it in the same way? More attention should be given to helping students develop problem-solving techniques that will be adequate in a wide range of problem situations. When a person meets a problem the author of this article says that he either (1) tends to avoid it, (2) applies only a limited stock of techniques to its solution, (3) is satisfied with a partial solution, (4) changes the problem completely, or (5) escapes from it entirely.

If this is true, how much does emotion influence critical thinking? This study reports the effect dogmatism has on critical thinking.

You will be interested to note that persons rated low in dogmatism were superior in critical thinking, had less errors, and were less likely to solve a problem with partial information. Perhaps as teachers we need to teach in such a way as to unleash the mind of the dogmatic and thus increase his power in problem solving.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The methods course

GORDON D. MOCK, *Western Illinois University, Macomb, Illinois.*

What topics should be studied in a methods course? History may help provide an answer.

THE PURPOSE OF THIS ARTICLE is to present background material on the methods course to see if some indications can be found as to how the subject might be improved in the future. In the process, the role that the high school teacher might play in this improvement is discussed.

By the turn of the century, roughly, the high school in the United States had passed through two stages. The first was the Latin Grammar School with its rather fixed curriculum designed for a single purpose. The second was the Academy, which operated on private funds but tried to present a broader program of studies. The new phase evolving was to be our free public high school.

When the high school reached this stage, the need for some sort of professional education for the high school teacher became evident. No longer would the college graduate with nothing else to do, the graduate student waiting for funds to continue with advanced work, or the unmarried female relative of a board member be available in the numbers needed or be adequate for teaching in the high school. The seeds of professional education were sown about the beginning of the nineteenth century and by the end of that century were beginning to sprout tender shoots.

Our particular topic of interest, the methods course, evidently first appeared in the latter part of the century. Since much of the early interest in professional education was centered about elementary-school teachers, and they were educated in the academies and normal schools, it is not surprising to find the first methods course work appearing in the latter type of institution. By the end of the century, courses on the teaching of algebra and geometry were being included in the program of education for high school teaching. The University of Michigan, which established such a course in 1893, was probably the pioneer in this area.

Note that the course introduced at Michigan, and shortly thereafter in various other institutions, was not called a methods course but a course on the teaching of algebra and geometry. The exact content of this type of course is difficult to ascertain. The content of what has come to be called the methods course has not been as uniform as, say, a course in college algebra.

To get an idea as to what has been discussed in the methods course, a study was made of the books most likely to be used as texts for such a course and of committee reports of a national character that might have served as supplementary material. This study yielded some indication as to the course content.

A composite of all this material would indicate the potential content of a maximal methods course, one that would discuss all of the topics presented in the reference material. The committee reports are mainly curriculum studies, but they also include a briefer treatment of the aims of mathematics education, of problems associated with the pupil (such as acceleration, testing, and psychological adjustment), and of programs for teacher preparation. The content of the texts has

been classified into twelve categories as follows: (1) values, (2) algebra, (3) geometry, (4) advanced high school mathematics, (5) general mathematics, (6) curriculum, (7) methods, (8) college mathematics, (9) evaluation, (10) psychological problems, (11) professionalism, and (12) visual aids.

A methods course may be thought of as a discussion of some of the topics mentioned, based for text or reference purposes on a committee report or a book dealing primarily with problems in the teaching of mathematics. In addition, there may be special reports, exercises, or lectures as the individual instructor determines. The difficulty lies in determining *what* topics to include, since there are more than could possibly be adequately covered in a three-semester-hour course.

The historical development of the materials assumed to be used in the methods course may offer some help in finding a temporary solution to the problem. One historical generalization that seems valid is that the number of topics discussed in the methods texts has increased. As new books appear, they generally retain all the old topics and add new ones. Early books did not discuss general mathematics, evaluation, or psychological problems to any extent. Advanced high school mathematics or college mathematics have been added in a few later texts. Visual aids, except in connection with geometry, were almost unknown. The curriculum was not a source of discussion to the extent it is today. On the other hand, the only topic that seems to have lost in interest concerns the discussion of various "methods" for presenting material, e.g., the lecture method or the heuristic method.

What seems to be happening is a general build-up of problems over a period of years. In a time when mathematics is an unpopular subject, an attempt is made to impress the methods student with the value of mathematical study, and then this remains as a topic in the methods course. When psychological problems come to the fore, we plan in mathematics education for individual differences, for proper transfer, and for *real* understanding, and this too remains as a topic in the methods course. Visual aids become a fad, and this topic is promptly added. And so for many others.

The only discussion that appears in every textbook that might be used as a methods text is on the subject matter of algebra and geometry. The discussion may include such things as showing how various ninth-grade topics are related, or how induction could lead to the discovery of a mathematical proof in a particular axiomatic system.

The point of this review is to indicate that the wisest choice of subject matter for a methods course is *not* determined by simple historical accretion or accident. There must be some basic philosophy concerning the content of the courses necessary for the future teacher, and this should determine the course syllabus. Not everyone will have the same beliefs, it hardly needs to be pointed out, but we need to generate and communicate some ideas in this area so that methods courses will be improved.

As a result of this study, it seems to me that we should try to do two things in an undergraduate course in methods: (1) briefly review the history of the teaching of mathematics in this country, and (2) discuss the actual content of high school mathematics as extensively as possible. This course would include material that is not likely to be covered in other courses, as well as content that is a matter of immediate necessity for the pre-service teacher.

As for the other topics, some are better discussed in other courses. Psychological problems should be discussed under competent instruction in a psychology class, and advanced topics in mathematics should be obtained in the appropriate class in mathematics. Some topics are too insignificant to take up the limited hours in a methods course—visual aids and how

to manage a mathematics club are examples. The discussion of the curriculum is one that will continue for the lifetime of the teacher. For the teacher with no experience, the value of much detail on curriculum problems per se seems questionable.

Possibly some of the people who would strongly agree or disagree with these opinions are practicing classroom teachers who have had a methods course. They could aid their "old methods prof" if they would contact him after a year or so of teaching and explain just what was of most value and what was of least value in their course. Obviously, complete unanimity will never be reached, but the communication hopefully may generate some new ideas, and such ideas are at the base of improved courses and hence of improved teaching.

What's new?

BOOKS

College

Algebras and Their Arithmetic, new edition, Leonard Eugene Dickson. New York: Dover Publications, Inc., 1960. Paper, xii+241 pp., $1.35.

Co-ordinate Geometry, new edition, Luther Pfahler Eisenhart. New York: Dover Publications, Inc., 1960. Paper, xi+298 pp., $1.65.

The Higher Arithmetic, An Introduction to the Theory of Numbers, H. Davenport. New York: Harper and Brothers, 1960. Paper, 172 pp., $1.35.

Introduction to Matrices and Linear Transformations, Daniel T. Finkbeiner, II. San Francisco: W. H. Freeman and Company, 1960. Cloth, vii+248 pp., $6.50.

An Introduction to the Theory of Numbers, Ivan Niven and Herbert S. Zuckerman. New York: John Wiley and Sons., Inc., 1960. Cloth, viii+250 pp., $6.25.

A Primer of Real Functions (Number 13, The Carus Mathematical Monographs), Ralph P. Boas, Jr. The Mathematical Association of America, 1960. Cloth, xiii+189 pp., $4.

Special Functions, Earl D. Rainville. New York: The Macmillan Company, 1960. Cloth, xii+365 pp., $11.75.

Statistics Manual, Edwin L. Crow, Frances A. Davis, and Margaret W. Maxfield. New York: Dover Publications, Inc., 1960. Paper, xvii+288 pp., $1.55.

Miscellaneous

Characteristics of Teachers, Their Description, Comparison, and Appraisal, David G. Ryans. Washington, D.C.: American Council on Education, 1960. Cloth, xxiii+416 pp., $7.50.

Free and Inexpensive Educational Aids, Thomas J. Pepe. New York: Dover Publications, Inc., 1960. Paper, 289 pp., $1.35.

Ingenious Mathematical Problems and Methods, L. A. Graham. New York: Dover Publications, Inc., 1959. Paper, 237 pp., $1.45.

Laboratories in the Classroom (New Horizons in Science Education), Science Materials Center, Inc., a division of The Library of Science. New York: Science Materials Center, Inc., 1960. Paper, 96 pp., $1.45.

The Nature of Science and Other Essays, David Greenwood. New York: Philosophical Library, 1960. Cloth, xiii+95 pp., $3.75.

The Restless Atom, Alfred Romer. Columbus, Ohio: Wesleyan University Press, 1960. Paper, 198 pp., 95¢.

BOOKLETS

High School

Algebra Problems, Donald S. Russell. New York: Barnes and Noble, Inc., 1960. Paper, vii+134 pp., $1.75.

Supplementary Units in Contemporary Arithmetic and Elementary Algebra, Edwin I. Stein. Princeton: D. Van Nostrand Company, Inc., 1960. Paper, 75 pp., $1.

College

A Survey of Basic Mathematics, A Text and Workbook for College Students, Fred W. Sparks. New York: McGraw-Hill Book Co., 1960. Paper, viii+257 pp., $3.95.

Bright seventh- and eighth-grade children challenge me

ELINOR B. FLAGG, *Illinois State Normal University, Normal, Illinois.*

"This can't be mathematics; it's so much fun."
"If we find another way to do square root, may we use it?"

THE IMPROVEMENT of the nation's educational opportunities is a continuing problem. In particular, the problem of improving opportunities for the academically talented child in the elementary school appears to be of major interest. Attesting to this statement is the following evidence: (1) the number of articles concerning the teaching of these children which have been appearing in professional periodicals and in daily and weekly news publications; (2) the frequency with which gifted-child discussions are listed in professional programs; (3) the increase in the number of school systems seeking to make some provision for the gifted child, and (4) the large number of observers who visit classes for the gifted child on our campus.

School authorities from California to the east coast are searching for, and trying out, programs which have as their special purpose the intellectual development of the gifted child. Illinois State Normal University, in its campus laboratory school, is trying out two such programs.

One of these programs, in existence since the school year 1955–56, includes children selected from the campus elementary laboratory school. The second program has been offered during the summer sessions of 1959 and 1960. For the summer program, children are selected from the schools of Bloomington and Normal, a different group participating each summer. None of the children in the summer groups has been enrolled in the campus school before coming into this special program.

Selection of participants has been essentially the same for the two programs. The children came from fifth and sixth grades. The original recommendation came from the classroom teacher; it was supported by results of tests, by achievement records, and by consideration of personal characteristics. Teachers recommended children who, in their judgment, would profit by the special programs. Principals and co-ordinators for Special Education also reviewed each case. The children were given an individual Stanford-Binet test by the University Testing Service. Results were interpreted by a committee composed of a representative of the testing service, co-ordinators for Special Education, principals, classroom teachers, and the teachers conducting the special classes. A minimum cutting score of 130 was set; the highest score was 163. Parents were consulted to determine whether they wished to have their children participate in the program.

It was agreed by those concerned that if at any time a child, or a parent, or a teacher, felt that the child was being overworked or was not profiting from the special program, he might withdraw from the class. From the time the campus group

was organized, a few children have dropped out, and a few have been added. The drop-outs occurred because of health or because child and parents felt that participation in other activities was more important to the development of that particular child. Additions to the group for the most part came from children newly enrolled in the campus school.

Each of the two special programs included instruction in three subject matter fields: social studies, science, and mathematics. Instructors were specialists in their fields; the same instructors participated in both programs. In each instance, the fifth and sixth graders met together as a single class unit.

The children participating from the campus school are partially segregated from their regular classroom. They meet two fifty-minute periods each week for special work in mathematics and also have two periods each week for social studies and two for science. Five of these six special sessions are scheduled at eight o'clock in the morning, before regular classroom work begins. One meeting is set at a time when the children miss one hour of work in their regular room. The children are required to keep up with their regular classroom assignments and activities; the special classes are the activities for which they must find extra time.

This group was organized during the 1955–56 school year with seven children from the fifth and sixth grades of one of the campus elementary laboratory schools; in the fall of 1956, the group was enlarged to fifteen children from the two campus elementary schools. This group, with a few drop-outs and a few additions, continued until the end of eighth grade. At the close of 1958–59, there were nine seventh graders and six eighth graders. The six eighth graders enrolled in a second-semester algebra class during the 1959 summer session at the University High School, and in the fall they enrolled in a geometry course in the same high school.

The children who became eighth graders in the fall of 1959 continued as a special group. In November, the high school mathematics department administered a standard end-of-first-semester test (Seattle Algebra Test), and all of the children ranked high by this measurement. In the spring of 1960, the high school administered two standard end-of-first-year tests: the Lankton Algebra Test, and the Co-operative Elementary Algebra Test. Eight of the nine children made scores sufficiently high for the high school to be satisfied that they could pass up the first-year algebra course. In September, they will enter a geometry course.

In the fall of 1959, a new seventh-grade class was selected for special work; in the fall of 1960, another seventh-grade group will be added. These two groups, since they will be in the special work program only two years, will miss some of the mathematics they would have had if they had started as fifth or sixth graders. Some of the same content will be included, however, and an effort will be made toward having these groups accomplish the learnings necessary for proficiency in freshman algebra.

For the summer program, twenty-one children enrolled in 1959; nineteen completed the summer's work. One child dropped out because of health and one dropped out because his family went on a long vacation. In 1960, again, nineteen children completed the summer program.

These children met as a segregated group each morning from eight-thirty to eleven; during this period they spent approximately forty-five minutes with each of three instructors (social studies, science, mathematics). At eleven o'clock they joined other fifth or sixth graders for physical education. There was no afternoon session.

Over-all aims are the same for both programs: (1) to give children opportunity for intellectual development through examination of topics and materials which present a challenge beyond that usually possible in regular classroom work; (2) to

give teachers-in-training opportunity to observe bright children at work as a group and to consider ways in which they might adapt some of these materials for their own use.

To implement these general aims, objectives for the work in mathematics (for both groups) were set up as follows: (1) increased interest in mathematics, (2) increased understanding of our number system, (3) appreciation of various number systems, (4) appreciation of forms used in present-day operations compared with those formerly used, and (5) appreciation of historical development. For the campus group, eventually a sixth objective was added: mastery of the algebra included in the high school algebra course specified for freshmen.

These objectives may appear to be the same as those used in a regular classroom, and perhaps they are also appropriate there. However, the learnings accomplished by the academically talented children go far beyond those of the children in the usual classroom. The gifted children have the intellectual capacity for deeper understanding, for greater insight, for making more observations and discoveries than average children. They can be "extended" beyond the reach of their graded text in arithmetic.

The mathematical content offered in both programs was selected with a view to carrying out the given objectives, and it was chosen from material which would not duplicate any of the arithmetic included in regular classroom work. Topics were selected which provided opportunity for teaching *real mathematics.* They were selected because they presented opportunity for going deeper into the reasoning inherent in mathematics; they provided a vehicle by means of which the children might become more resourceful in mathematical thinking and less dependent on rote learning than the average child; they provided a means of seeing the "why" of motions to which they had become accustomed.

Topics which have been used are: (1) algorithms for performing addition, subtraction, multiplication, and division other than the traditional forms, with emphasis on place value and on meanings of operations, (2) ways of recording and writing number symbols different from our own, (3) the Hindu-Arabic number system, (4) numeration with bases other than 10, (5) the metric system, (6) scientific notation, (7) prime numbers and factors, (8) tests for divisibility, (9) introduction to sets and sentences, and (10) equalities and inequalities.

To enlarge somewhat on one of these topics, the following explanatory statement is offered: In an attempt to interpret our decimal notation in a meaningful way, study was begun by analyzing place value. Each child was provided with an abacus, and a homemade counting board was devised. Dried lima beans were used as counters. There was discussion of the place value of the counters as they were placed on the counting board, and the children not only talked about "carrying" and "borrowing" but they actually did the "carrying" and the "borrowing." They also learned to analyze, or "break down," the place value of the symbols in the Hindu-Arabic system. This gave opportunity for the introduction of exponents, as powers of 10, and their use in analyzing place value.

Following the study of the Hindu-Arabic notation, the children were led to consideration of notations using bases other than 10. In both programs, the children found this creative and inventive work extremely interesting and challenging; they "discovered" that any number might be used as a base; they wrote numerals for other bases, made tables of addition and multiplication facts for the "new" symbols, learned to add and to multiply, and performed some subtraction.

A few comments and questions which have come from the children participating may be of interest; they give evidence of

curiosity, of thinking, and of creativeness; they also reveal some of the content with which the children worked.

One boy remarked, "I was in the fifth grade before I knew why 10 was the first two-digit number we write; it's because we've agreed that the first two-digit number must show what base we are using." Following this, a sixth-grade girl said, "It's the same for the other bases; for base 2, we write 10, the name of the base." From another fifth grader: "Now I'll know what to tell my children when they come home from school and ask me why our number system is this way. Part of the reason is that we've agreed to have it that way, and part of it is that some things grow out of the laws that we have."

"What do you do when the base of a number system is larger than 10?" The answer came from one of the children, "You have to make up more symbols, more one-digit symbols, until you reach the name of the base." Another child added, "And when you write 10, it stands for the name of the base that you have chosen."

"Going from 22 to 100 in base 3 is just like going from 99 to 100 in base 10."

"If we used base 5, I'd be old enough to drive."

"Yesterday we learned how to add signed numbers. I think I've found out how to subtract. I used a number line, and I think it works."

"Can a negative number be an exponent?" "Can a fraction be an exponent?"

"Can you multiply with Roman numerals?"

"If we find another way to do square root, may we use it?"

"This can't be mathematics; it's so much fun."

Literature concerning programs for the gifted child usually suggests that such programs fall into categories. Frequently these categories are listed as acceleration, segregation, and enrichment, with a fourth category added which is described as "a combination of the first three." If the programs at Normal can be put into any of these compartments, they fall more nearly into that of enrichment than any other. However, there is acceleration to the extent that the eighth graders, thus far, have passed up the algebra course in high school. And there is some segregation, as has already been described. Perhaps, then, Normal's programs fall into the fourth category: a combination of the first three.

The term *enrichment* must be given further comment. It is used occasionally to mean "horizontal" enrichment, and occasionally to mean "vertical" enrichment. Vertical enrichment is interpreted as "extra topics which are in advance of the grade level of the child," and horizontal enrichment as "extra topics which are within the grade level of the child." For the most part, enrichment in the programs which have been described has been of the horizontal variety; that is, it has been within the framework of the child's already established learnings.

But there has been some of the vertical variety. This is true with respect to the topic of algebra. Again, it is true with respect to the teaching of exponents to fifth and sixth graders. Perhaps it might be said that the teaching of inequalities is still another instance of vertical enrichment.

Whatever name is attached to a program for the academically talented child, the reason for its existence appears to be the same. Each program has been planned for the purpose of challenging the child to intellectual development beyond that for which the regular classroom makes provision (1) by encouraging him to explore and to wonder, and (2) by guiding him to an appreciation of the logical, systematic structure of mathematics.

The instructors conducting the two special programs at Normal have reached a few conclusions concerning academically talented children. Like any other conclusions involving human beings, they are, of course, subject to modification, and prob-

ably they are not unique with respect to the two programs being discussed. At the moment, the following conclusions seem valid: (1) Academically talented children, like any children, have their individual differences; they learn at different rates, and achievement is not the same for every child. (2) They enjoy mathematical thinking; logical structure appeals to them; they like "if-then" thinking. (The elation and satisfaction shown by the children when they found they could add numbers in "new" bases is one bit of evidence to support this conclusion.) (3) They generalize easily. Frequently they accomplish a "learning" from a single illustration. (4) They need little drill, but occasionally there is a child who needs some form of practice. (5) Bright children, like any children, get into mischief if given the opportunity.

These children are a continuing challenge to their instructors. There is an urgent and ever-present need for providing a mathematical diet of such substance that doors to the future will be kept open; there is need for a mathematical content other than that of rote learning; there is need for a program which will inspire real thinking.

There is evidence that these programs have been challenging to the children; at least, they have been challenging to some children to some degree. One child said: "The work in mathematics helped us to apply things we already knew." Pressed for a more specific statement, she added, "Well, we already knew something about base 10, and we used that to help us with other systems." This child saw not merely a repetition of old learnings; she saw an extension of those ideas; she had been challenged to some rather original thinking.

The child who said, "That's neat," when the symbol $n^{1/2}$ was being discussed, added a moment later, "Everything fits together." He was asked to show what he had in mind. In his reply, he reminded his classmates that n^2 times n^3 is written as n^5; exponents are added. "And so," he said, "it is reasonable to add fractional exponents when numbers are being multiplied. If you multiply $n^{1/2}$ by $n^{1/2}$, you would write n^1, or just plain n. This shows that one of the two equal factors of n is $n^{1/2}$, and that makes it the square root of n."

These children have grown intellectually through participation in the special classes. But have they been challenged to their highest potential? Have they the ability to penetrate still further into mathematical understandings? If a systematic development is followed, it seems reasonable to assume many other challenging learnings might be accomplished.

This presentation has included a description of the administrative organization of two programs for academically talented children; it has included the method of selection of the children, some of the content used in the special classes, and reactions from a few of the children. These details have been presented with the thought that they may suggest ideas adaptable in other schools. Experience with classes of bright children is indeed gratifying, both to children and to teachers. It is the aspiration of those who have tried it, that others may have the same worthwhile opportunities.

Bibliography

Denny, Robert. *How to Challenge the Gifted in Arithmetic.* Des Moines, Iowa: The Denny Press, 1959.

Grossnickle, F. E. "Arithmetic for Those Who Excel," *The Arithmetic Teacher,* III (March 1956), 41–48.

Henry, N. B. (ed.). *Education for the Gifted.* Fifty-seventh Yearbook of the National Society for the Study of Education. Chicago: University of Chicago Press, 1958.

Hlavaty, J. H. (ed.). *Mathematics for the Academically Talented Student in the Secondary School.* Washington, D.C.: National Education Association and National Council of Teachers of Mathematics, 1959.

Kough, Jack. *Practical Programs for the Gifted.* Chicago: Science Research Associates, 1960.

Witty, Paul A. (ed.). *The Gifted Child.* American Association for Gifted Children. Boston: D. C. Heath and Company, 1951.

The fundamental theorem on linear programming

SHMUEL AVITAL, *Technical High School of the Israel Institute of Technology, Haifa, Israel.*

An expression ax+by *defined over a closed convex polygon assumes its maximum and minimum value at one (or more) of the vertices of the polygon.*

THE MARCH 1960 issue of THE MATHEMATICS TEACHER contains an article by Donovan Lichtenberg and Marilyn Zweng on linear programming problems for first-year algebra. The authors stress that no attempt was made to prove the fundamental theorem.

I think that every attempt should be made to place algebra on a deductive basis early in high school, and it seems to me that the following approach is within the grasp of the brighter pupils. I am omitting the introduction and the part leading to students' discovery, as those have already been beautifully developed in the aforementioned article.

The one-to-one correspondence between a set of ordered pairs and a set of points in a plane is assumed as known.

Given an expression of the form $ax+by$ defined over a set of ordered pairs (x, y), we shall try to find conditions for the pair (x_m, y_m) for which the expression will receive its maximum and minimum values. The following is a simplification and methodological reorganization of the usual proof for linear functionals over a closed convex polygon.

THEOREM 1: Given two ordered pairs (x_1,y_1) (x_2, y_2), the graphs of which are points $A_1(x_1,y_1)$ $A_2(x_2,y_2)$. Every ordered pair (x_0, y_0), the graph of which lies on the finite line A_1A_2, will have co-ordinates $[tx_2+(1-t)x_1, \quad ty_2+(1-t)y_1]$, where $0\leq t\leq 1$.

Proof: Assume

$$\frac{\overline{A_1A_0}}{\overline{A_1A_2}}=t$$

$$\Delta A_0A_2M\sim\Delta A_1A_2N.$$

Hence

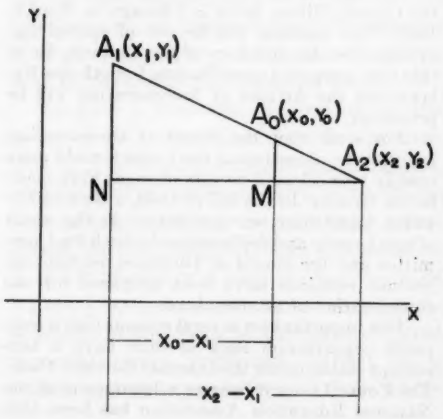
$$\frac{x_0-x_1}{x_2-x_1}=t$$

$$x_0-x_1=tx_2-tx_1$$

$$x_0=tx_2+(1-t)x_1.$$

In a similar way, by drawing a parallel to the X axis through the point A_0, we prove

Figure 1



$y_0 = ty_2 + (1-t)y_1$. As A_0 lies on the finite line A_1A_2, We have

$$0 \leq \frac{\overline{A_1A_0}}{\overline{A_1A_2}} \leq 1.$$

Hence the inequality $0 \leq t \leq 1$ is assured.

Note: The converse is true too but is not needed in the following.

THEOREM 2: Given an expression $ax+by$ defined over a set of ordered pairs, the graph of which, in the plane, is the line with end points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$. The value of the expression for any point on A_1A_2 lies between the end-point values.

Proof: By Theorem 1, any point A_0 on A_1A_2 has ordinates $(tx_2+(1-t)x_1, ty_2+(1-t)y_1)$. Let the value for A_1 be $m_1 = ax_1+by_1$, and the value for A_2 be $m_2 = ax_2+by_2$. The value for A_0 will be

$$\begin{aligned} m_0 &= a[tx_2+(1-t)x_1]+b[ty_2+(1-t)y_1] \\ &= t(ax_2+by_2)+(1-t)(ax_1+by_1) \\ &= ax_1+by_1+t[(ax_2+by_2)-(ax_1+by_1)] \\ &= m_1+t(m_2-m_1). \end{aligned}$$

As $0 \leq t \leq 1$, m_0 lies between m_1 and m_2.

Corollary: The value of an expression $ax+by$, defined over the boundary of a closed polygon, for any point on this boundary, lies between its values for the corner points of the boundary.

THEOREM 3: The value of an expression $ax+by$ defined over a set of ordered pairs, the graph of which is a closed convex polygon, for any point inside the polygon, lies between the values it assumes for some points on the boundary of the polygon.

Proof: Let P be any point inside the polygon. As the polygon is convex and closed, any line through P cuts the boundary at two points and all the points of the segment joining these two points belong to the polygon. Therefore the expression $ax+by$ is defined for all the points of this line. Hence by Theorem 2 the value for P lies between the values for the end points that lie on the boundary.

THEOREM 4 (The Fundamental Theorem): An expression $ax+by$ defined over a closed convex polygon assumes its maximum and minimum value at one (or more) of the corner points of the polygon.

Proof: By Theorem 3 the value of the expression for any point inside the polygon lies between the values for some two points on the boundary. Again, by the corollary to Theorem 2, the values for the boundary points lie between the values for the corner points. As the number of the corner points is finite, hence for one (or more) of them the value is a maximum and for another (or more) it is a minimum.

Notice of Annual Business Meeting

The Annual Business Meeting will be held at the Conrad Hilton Hotel in Chicago on April 6, 1961. This meeting will be one of special significance to the members of the Council, for at this time proposed amendments to both the Bylaws and the Articles of Incorporation will be presented.

For some time the Board of Directors has felt that the objectives of the Council could more readily be realized if certain changes were made in the Bylaws. In the fall of 1959, a Bylaws Revision Committee was appointed. As the result of much study and deliberation by both the Committee and the Board of Directors, certain important revisions have been prepared for the consideration of the members.

It is important for several reasons that a nonprofit organization such as ours have a tax-exempt status under the Internal Revenue Code. The Council since it became a department of the National Education Association has been able to take advantage of the latter's tax exemption. However, it is questionable whether this practice could be adequately defended before the courts if it were challenged. So that the Council may apply for its own independent tax exemption, certain amendments of the Articles of Incorporation are required. These amendments have been worked out in consultation with an attorney and have been approved by the Board of Directors.

A statement of the proposed amendments both of the Bylaws and the Articles of Incorporation will be mailed from the Central Office to each member of the Council on February 27. It is hoped that every member will study these materials thoughtfully and that a large group of members will be present to discuss and make decisions on them at the Annual Business Meeting on April 6.

M. H. AHRENDT, Executive Secretary

• EXPERIMENTAL PROGRAMS

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

Mathematics in transition[1]

by Julius H. Hlavaty, Director, Mathematics Commission Program

The Commission on Mathematics, consisting of university mathematicians, leaders in the training of teachers of mathematics, and secondary school teachers, was organized in 1955.

This group spent four busy years formulating tentative proposals, discussing them, tearing them apart, rejecting some, reformulating others, and elaborating still others. Not only did each proposal have to survive the gamut of the criticisms and reactions of the individual members of the Commission, but each was made the subject of careful and critical review by the profession at large. Representatives of the Commission presented the developing program of the Commission to innumerable local, regional, state, and national conferences of teachers of mathematics, mathematicians, and educators. The reactions of the profession at large had a marked influence on the final report.

Nine-point program proposed

The Commission succeeded in formulating and proposing a nine-point program for college-capable students:

1. Strong preparation, *both* in concepts *and* in skills for college mathematics at the level of calculus and analytic geometry.
2. Understanding of the nature and role of deductive reasoning—in algebra, as well as in geometry.
3. Appreciation of mathematical structure ("patterns")—for example, properties of natural, rational, real, and complex numbers.
4. Judicious use of unifying ideas—sets, variables, functions, and relations.
5. Treatment of inequalities along with equations.
6. Incorporation with plane geometry of some co-ordinate geometry, and essentials of solid geometry and space perception.
7. Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers.
8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular).
9. Recommendation of additional, alternative units for grade 12: *either* introductory probability with statistical applications *or* an introduction to modern algebra.

This nine-point program was elaborated in the first volume of the Report of the Commission on Mathematics, *Program for College Preparatory Mathematics*, which was published in the spring of 1959.

To give a concrete illustration of the point of view that should guide the creation of a new curriculum in mathematics, and to provide teachers with some of the new subject-matter material which is proposed in the program, the Commission found it necessary during the four years of its work to issue a number of small publications in the form of pamphlets. These materials were improved, others of a similar nature were written, and the whole

[1] Reprinted in part with permission from *College Board Review*, No. 41, Spring, 1960.

was incorporated in the second volume of the Report, the *Appendices*.

During the four years of its work, the Commission had accumulated a mailing list of 13,000 interested people—teachers, students, mathematicians, and parents. During the period of preparation of the Report, these interested people had absorbed hundreds of thousands of pamphlets, the piece-meal preliminary products of the Commission's work in progress.

The first free distribution of *Program for College Preparatory Mathematics* reached some 39,000 readers, including the Commission's mailing list, the membership of the principal professional organizations, the American Mathematical Society, the Mathematical Association of America, the National Council of Teachers of Mathematics, and thousands of school superintendents. Since that time the demand for both volumes of the Report has been large and steady.

The interest in the Report of the Commission has been country wide and world wide. The Commission receives a steady stream of questions, comments, and requests from every state in the Union and from such far places as Nigeria, Japan, Poland, Israel, Australia, Sweden, Thailand, the Philippine Islands, and South America. There is every indication that the work of the Commission will have profound effects on the secondary-school curriculum. This is evidenced by the character of the correspondence which is received by the Commission.

In the first place, there are continuing requests for the publications of the Commission—even those that are out of print. It is clear that the wave of unprecedented interest and enthusiasm for reforming mathematical instruction is reaching segments of the professional and lay world that were never involved before; among them groups of administrators at the secondary-school level and interested parents. A particularly striking illustration of this was the permission requested by the Inter-American Statistical Institute of the Pan-American Union to translate into Spanish the Commission's *Introductory Probability and Statistical Inference*. The permission was, of course, granted.

These important new audiences are first of all interested essentially in the generalities of the Commission's work: the reasons for the need for reform; the problems of high school mathematics; and the general tenor of the Commission's recommendations.

In addition, many individual teachers, school systems, and textbook writers are appraising the specific implications of the Report. It seems certain that this scrutiny will bear significant fruits. The hectic and sometimes hysterical reaction to Sputnik and the consequent public appraisal of school science and mathematics programs has settled down to a plodding and stubborn reconstruction of the mathematics curriculum. Individual teachers who had experimented with the early publications of the Commission (*Sets, relations, and functions; Introduction to algebra;* and so on[2]) are now ambitiously rewriting their courses in mathematics when they are in a situation where they are free to do so. Groups of teachers who used the Commission's material in the summer institutes organized and financed by the National Science Foundation are the cores of committees which are rewriting syllabi for whole school systems. Writers of established, successful texts, as well as those hopeful of writing the texts of the future, are knuckling down to the task of translating the specific recommendations of the Commission into textbooks.

This category of correspondents reads Chapter 4 of the Report ("Organization: a proposed sequence for the Commission's program") with a microscope. They ask searching questions; they point out missing commas. They want to know, "What do you mean by this?" They also read the *Appendices* carefully and freely adapt (and sometimes improve) the material. In

[2] Although out of print as pamphlets, these publications have been incorporated into the *Appendices*.

doing this, of course, they carry out the very intentions of the Commission. But most of all, they ask, "Is this what you mean?" and "How do we know this will produce better results than the traditional course of study?"

An increasing segment of persons responding to the Report is made up of tough-minded administrators who are working on the implementation of a new program in mathematics. Local school boards and state departments of education have strengthened, or in some cases initiated, programs of supervision in mathematics. They have been brought face to face with the concrete problems of teacher training and retraining in service: organizing and subsidizing in-service programs and finding people to conduct these training courses. Often they ask the Commission for speakers and instructors. State departments of education are reviewing their certification requirements, and they also ask for advice. It is clear that the fifth chapter of the Commission's Report ("Implementation: the vital role of teacher education") is becoming increasingly pertinent for all concerned, from individual teachers to state-wide systems. The schools of education are undertaking serious reviews of their programs for future teachers and for teachers now in service. The growing program of the National Science Foundation is a part of this major activity in teacher education, and recently, the American Association for the Advancement of Science decided to use Chapter 5 of the Report in its regional conferences on the Teacher Preparation-Certification Study.

Consensus emerging

In the beginning stages of this experiment in mathematical education it was easy to distinguish the role of the Commission from that of the quite different but contemporaneous Advanced Placement Program and the Illinois Plan. The Commission assumed an important position at the yearly summer and in-service training institutes through the use of its pamphlets and particularly of its *Introductory Probability and Statistical Inference*,[3] an experimental text used and reviewed very extensively, perhaps because they were the only materials available.

Today the role of the Commission is still quite distinctive and an important source of the swelling stream fed also by the massive work of the School Mathematics Study Group, the growing influence of the Illinois Plan, and the innumerable attempts sponsored by local, state, and federal bodies. The Report gave explicit and concrete form to what has emerged as a consensus of the many seemingly different approaches to the problem of a new curriculum in mathematics. This conclusion was underlined by the publication of *The Secondary Mathematics Curriculum*, the Report of the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics, in May 1959. While the National Council's recommendations do not agree in every detail with those of the Commission, they do agree in major outline and emphasis.[4]

The School Mathematics Study Group (Yale Project) accepted in essence the recommendations of the Commission. In its tremendously productive writing project in the summer of 1959, the SMSG produced sample textbooks for grades 7 through 12. In only one instance—geometry—did it significantly depart from the recommendations of the Commission. In this case it produced an up-dated and relatively rigorous textbook in Euclidean

[3] (New York: College Entrance Examination Board, 1957).

[4] In algebra, the SSCC stresses the importance of insight and understanding, the importance of pattern and structure, the role of the laws of a field, some utilization of vocabulary and of the concepts of sets, and the study of inequalities as well as equations. In geometry, while it gives more place to synthetic Euclidean geometry than does the Commission's program, it recommends some solid geometry in the year's work. It urges that co-ordinate geometry be begun in the ninth year and be extended in the eleventh year. There is substantial agreement between the two reports insofar as the work of the eleventh and twelfth years is concerned.

geometry, which however includes a treatment of solid geometry and a unit on coordinate geometry. In all other areas it strove deliberately to implement Commission's recommendations.

Possible results

In the present auspicious atmosphere for curricular reform, the Commission's work may prove to be a guide to developments in areas beyond that of its original assignment, the preparation of a program for college preparatory mathematics.

One possibility, if the Commission's program is realized to any significant degree at the secondary level, is that the undergraduate and graduate programs in mathematics at the colleges and universities will be deeply affected. A college mathematics department which grew accustomed to and had some confidence in the traditional definition of requirements (algebra through quadratics, for example), must learn to understand the new requirements and must build its own program based on them. The Committee on the Undergraduate Program of the Mathematical Association of America, is already studying this problem.

Another area which has excited schoolmen in recent years is the problem of providing a challenging program for the more able students. While the Commission's program, in effect, is a minimal program for the talented, it provides, through the variety of optional topics, through the *Appendices*, and through the varied program for the twelfth year a storehouse of possible enrichments. In practice, there is some confusion between the Advanced Placement Program and the various alternatives suggested for the twelfth year. There is room for study, investigation, experimentation, and clarification in the area of the superior student.

A third area of concern in mathematical education is the whole elementary-school program. An ideal program in the secondary schools presupposes a sound program in elementary instruction. What should be the contents of such a program, and how will it be realized? What are the implications in the area of teacher training and retraining? Major experimental programs such as the University of Illinois Arithmetic Project and the Greater Cleveland Mathematics Plan have begun, and the School Mathematics Study Group and the National Council of Teachers of Mathematics are readying a major assault on the problem.

Bearing in mind the American commitment to universal secondary education, even a definition of the college preparatory program is only an approach to the solution of the problems represented by the mathematical education of all American youth. Can the experience and the techniques of the Commission on Mathematics be utilized helpfully to attack this important problem of secondary education?

In assigning the Commission on Mathematics its task, the College Board ventured into an area of delicate questions. In the bad old days, the colleges had dictated to the high schools and dominated their curricula. But the Commission was organized to investigate, and if possible, improve a situation of common concern to the high schools and the colleges. Through its representative membership and cooperative working methods, the Commission has attempted to express the best thinking of the mathematical community. Now that the Report has been submitted and warmly received, it is sometimes asked whether the College Board intends to legislate changes in the mathematical curriculum by changing its tests. The answer is "No." The Commission's recommendations are "being taken as a guide for a gradual change in the examinations. This change should extend over a period of years and will proceed as mathematics programs actually being taught in the schools are changed."[5]

[5] Report of the Commission on Mathematics, *Program for College Preparatory Mathematics*, p. 61.

The Greater Cleveland Mathematics Program

by George H. Baird, Executive Director, Educational Research Council of Greater Cleveland, Cleveland, Ohio

The Greater Cleveland Mathematics Program (GCMP) was developed in 1959-60 through the facilities of the Educational Research Council of Greater Cleveland in response to a concerted request by the superintendents of 21 school districts of suburban Cleveland. These superintendents had recognized that there was much research being done in the field of mathematics but that very little of the research results were actually being implemented in the classroom; this they wished to see accomplished.

The first stage of GCMP was designed primarily as a teacher-training program. It was realized that before new content could be presented to pupils, the teachers themselves would require additional training.

The superintendents of the participating school districts provided released time and in-service credit for the teachers who were assigned to attend the training sessions conducted by Dr. B. H. Gundlach of the GCMP. Specifically, intensive training during the 1959–60 school term was provided for the teachers of grade one; the other elementary-school teachers (2–6) received supplementary lectures and enrichment materials throughout the school year.

Actual classroom implementation of the GCMP material was begun in January, 1960, and as the results of a testing program conducted in May, 1960 indicate, the program caused a significant increase of 3.1 months in the arithmetic grade-placement for the experimental first-grade pupils.

During the 1960–61 school year, the training is being extended to include second- and third-grade teachers with the teachers of grades 4, 5, and 6 receiving supplemental lectures (oral and TV) and written enrichment materials. An intensive training program by Dr. Jack Forbes for the teachers of grades 7, 8, and 9 will also be undertaken; the program for the teachers of grades 10, 11, and 12 will be enrichment-orientated. The training programs for both the elementary- and secondary-school teachers will be conducted through the use of television, lecture-discussion sessions, and classroom demonstrations. The local commercial station KYW-TV, Westinghouse Broadcasting Company, is providing one-half hour each day for the viewing of video-taped lessons. In addition, biweekly or monthly lecture-discussion sessions will be conducted, with teachers being given released time by the superintendents.

In addition to the introduction of the regular GCMP content, the experimental use of the School Mathematics Study Group's (Yale University) new materials for grades 4, 5, and 6 will be tried in 60 selected classrooms of the 21 Greater Cleveland school systems involved in this program.

The schools involved in this program have a total enrollment of over 200,000 pupils.

The following persons serve as mathematics consultants to the program: Drs. Max Beberman, Charles F. Brumfiel, Jack E. Forbes, Bernard H. Gundlach, Phillip S. Jones, and John G. Kemeny.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Benjamin Banneker, pioneer Negro mathematician

by T. F. Mulcrone, S.J., Jesuit High School, Tampa, Florida

In almost every history of the Negro in the United States one can find an account of the incidental contributions of Benjamin Banneker (1731–1806) to the social history of his race. But no chronicle is readily available of the scientific life of Banneker as a student of mathematics, almanac compiler, surveyor, and astronomer. The object of this article is to provide such an account of the scientific activity of Benjamin Banneker, whom W. Douglas Brown called "the first American Negro to challenge the world by the independent power of his intellect," and to indicate how close Banneker comes to approximating the composite picture of the mathematician in the United States at the end of the eighteenth century.

Benjamin Banneker was born on a farm near the Patapsco River, about ten miles from Baltimore, Maryland, at what is now Ellicott City, Maryland, on November 9, 1731. His mother and father were Negroes—his mother free, his father a slave; his maternal grandmother was a white woman of English birth who was legally married to a native African, the son of an African king. In the early eighteenth century, a Negro of Banneker's class enjoyed most of the privileges of white men. James McHenry, [2]* who had been a secretary of President George Washington, a commissioner and signer of the Constitution of the United States, and a Senator from Maryland, tells us, in a letter of recommendation for Banneker, that the boy's parents "were enabled to send him to an obscure school, where he learned, when he was a boy, reading, and arithmetic as far as double fractions." This school numbered a few white and two or three Negro children taught by the same white schoolmaster. The school was open only during the winter season, and Banneker ceased attending it after he had grown big enough to assist his father on the farm. He was not given to games, but found genuine delight in his books. Beyond these very meager rudiments he was his own teacher, keeping up, of his own initiative, his intellectual interests all through life.

His livelihood was earned principally in the laborious cultivation of a farm, a necessity that limited very drastically the intellectual and scientific labors to which he was inclined and to which, it would seem, he would have dedicated himself with possibly substantial achievement had he enjoyed the opportunity. McHenry testified, in the letter mentioned above, to the need that compelled Banneker

> . . . to struggle incessantly against want in no way favorable to improvement. What he learned, however, he did not forget; for as some hours of leisure will occur in the most toilsome life, he availed himself of them—not to read and acquire knowledge from writings of genius and

* Numbers in brackets refer to the Notes and References at the end of the article.

discovery, for of such he had none, but to digest and apply, as occasion presented, the few principles of the few rules of arithmetic which he had been taught in school.

This kind of mental exercise formed his chief amusement, and soon gave him a facility in calculation that was often serviceable to his neighbors.

During his early manhood Banneker also found great delight in mechanics, and at this time of his life he constructed, entirely from hardwood and with no other model than a borrowed watch, a clock that served as a reliable timepiece for over twenty years.

It was his interest in things mechanical and his ability in arithmetic that brought Banneker to the attention of the Ellicott brothers, who, in 1772, settled in the vicinity of his farm and began the construction of the flour mills that were the beginning of the village of Ellicott's Mills, the present Ellicott City, Maryland. Banneker frequently visited the mills during their construction. This Quaker family, destined to become a leading Maryland family, exercised great influence on Banneker, and was instrumental in fostering his career.

Banneker early won the reputation of being the smartest mathematician for miles around, skilled in solving mathematical puzzles quickly as well as accurately. Mrs. Martha Ellicott Tyson, his principal biographer, to whom I am indebted for most of the materials used in this sketch [1], informs us that Banneker

> ... was an adept in the solution of difficult mathematical problems. They were in his day, much more so than at present, the pastime of persons of culture. (See [5], pp. 85–89; [6].) Such questions were frequently sent to him by scholars in different parts of the country who wished to test his capacity. He never failed to return a solution, sometimes accompanying it by questions in rhyme of his own composition.

Thereupon Mrs. Tyson quotes from a letter, written in November 1752, of Charles W. Dorsey, who was a clerk in the store of Ellicott & Company at the time and knew Banneker personally:

> He was fond of, and well qualified to work out, abstruse questions in arithmetic. I remember he brought to the store one which he had composed himself and presented to George Ellicott [the father of Mrs. Tyson] [7] for solution. I had a copy, which I have since lost, but the character and deportment of the man were so wholly different from anything I had ever seen in one of his color; his question made so deep an impression on my mind that I have ever since retained a perfect recollection of it, except two lines which do not alter the sense.
>
> The following is the question:
> A cooper and vintner sat down for a talk,
> Both being so groggy that neither could walk;
> Says cooper to vintner, "I'm the first of my trade,
> There's no kind of vessel but what I have made,
> And of any shape, sir, just what you will,
> And of any size, sir, from a tun to a gill."
> "Then," says the vintner, "you're the man for me.
> Make me a vessel, if we can agree.
> The top and bottom diameters define,
> To bear that proportion as fifteen to nine,
> Thirty-five inches are just what I crave,
> No more and no less in the depth will I have;
> Just thirty-nine gallons this vessel must hold,
> Then I will reward you with silver or gold,—
> Give me your promise, my honest old friend."
> "I'll make it to-morrow, that you may depend!"
> So, the next day, the cooper, his work to discharge,
> Soon made the new vessel, but made it too large;
> He took out some staves, which made it too small,
> And then cursed the vessel, the vintner, and all.
> He beat on his breast, "By the powers" he swore
> He would never work at his trade any more.
> Now, my worthy friend, find out if you can,
> The vessel's dimensions, and comfort the man!
>
> Benjamin Banneker

Banneker never married. He took care of his mother until her death, his father having died in the meantime. When she died, the family property was divided between him and his three sisters. To secure the leisure necessary to devote himself to his studies, Banneker tried various expedients with different tenants, but nothing worked out satisfactorily. Finally he entered into agreement with Ellicott & Company, giving them full rights to his property after his death, in return for a

life-estate therein, an annuity of £12 a year, and enough land for cultivation as would give him some needed exercise. At the time of the agreement, Banneker figured he would live only fifteen years more. Actually he lived eight years longer than the time he had calculated. "This miscalculation," remarked his biographer Baker, "is said to have been the only mistake in mathematics Banneker ever made." (See [4], p. 113.)

Many were the favors which Banneker received from the Ellicotts, but the greatest of them was the loan of the books and instruments that so greatly influenced this Negro's subsequent life. In 1787, Banneker received from George Ellicott copies of Mayer's *Tables*, Ferguson's *Astronomy*, and Leadbetter's *Lunar Tables*, together with some astronomical instruments. At once, and without any help, he undertook the mastery of these works and instruments. Two years later he was able to predict a solar eclipse with considerable accuracy. Thereupon he determined to master—at the age of 58—the science of astronomy, the subject that was to remain his principal interest the rest of his life.

Some indication of the difficulties he encountered may be gathered from the following passage of a letter to George Ellicott, written October 13, 1789. Mrs. Anne T. Kirk, daughter of Mrs. Tyson, remarked that

> . . . it is thought proper to retain the original style of expression and erroneous spelling as the unlettered author, Banneker, left them.
>
> "In the Letter Concerning the number of Eclipses," wrote Banneker, "tho' according to authors the Edge of the penumbra only touches the Suns Limb in that Eclips, that I left out of the number—which happens April 14th day, at 37 minutes past 7 o'clock in the morning, and is the first we shall have; but since you wrote to me, I drew in the Equations of the Node, which will cause a small Solar Defet, but as I did not intend to publish I was not so very peticular, as I should have been, but was more intent upon the true method of projecting a Solar Eclips. It is an easy matter for us, when a Diagram is laid down before us, to draw one in resemblance of it, but it is a hard matter for young Tyroes in Astronomy, when only the elements for the projection is laid down before him, to draw his Diagram with any degree of Certainty."

After Banneker's death two of those books (which he directed be returned to George Ellicott) became the property of Mrs. Kirk. Of these volumes she wrote:

> One of the works is upon surveying and gives evidence of much careful use. The other is a large volume entitled *Tabulae Solis et Lunae* by Tobias Mayer, ... This book has evidently been of great value to our student. The paper in some places is quite worn away. As much as the text is in Latin, it appears that Banneker must have had some knowledge of that language.

Several of Banneker's extant letters deal with corrections in the sources he used. For example, to George Ellicott he wrote:

> Errors that ought to be corrected in my astronomical tables are these:—2d vol. Leadbetter, p. 204, when ♄ anomaly is 4^s 30° the equation 3° 30′ 4″ ought to have been 3° 28′ 41″. In ♂ equation p. 135, the logarithm of his distance from ☉ ought to have been 6 in the second place from the index, instead of 7, that is, from the time that its anomaly is 3^s 24° until it is 4^s 0°. (See [3], p. 82.)

And in such corrections as the following, also addressed to George Ellicott, we find evidence of genuine scientific spirit:

> It appears to me that the wisest men may at times be in error; for instance, Dr. Ferguson informs us that, when the sun is within 12° of either node at the time of full, the moon will be eclipsed; but I find that, according to his method of projecting a lunar eclipse, there will be none of the above elements, and yet the sun is within 11° 46′ 11″ of the moon's ascending node. But the moon, being in her apogee, presents the appearance of this eclipse. (See [3], p. 82.)

On February 2, 1791, Major Andrew Ellicott [8], the eldest son of Joseph Ellicott, one of the brothers who founded Ellicott's Mills, was commissioned to survey and lay out what is now the District of Columbia, while Thomas Johnson, David Stuart, and Daniel Carroll were appointed to the Commission. Shortly afterwards, Major Pierre C. L'Enfant was appointed to assist Ellicott. Presently, at the request of Ellicott, and also, it has been suggested, on the recommendation of Secretary of State Thomas Jefferson's friends in the American Philosophical Society, Ban-

neker was nominated by Jefferson as an assistant to the Commission and President George Washington made the appointment.

On August 19, 1791, Banneker wrote to Jefferson the celebrated letter in which he pleads for the rights of the thousands of slaves who were held in the Colonies. Together with that letter Banneker sent a copy of his first almanac, that of the year 1792, which he had prepared, at the urging of George Ellicott, from his own calculations. This letter contains the following words, so eloquent of Banneker's love of scientific inquiry.

> This calculation, Sir, is the production of my arduous study, in this my advanced stage of life; for having long and unbounded desire to become acquainted with the secrets of nature, I have had to gratify my curiosity therein, thro' my own assiduous application to astronomical study, in which I need not recount to you the many difficulties and disadvantages I have had to encounter.

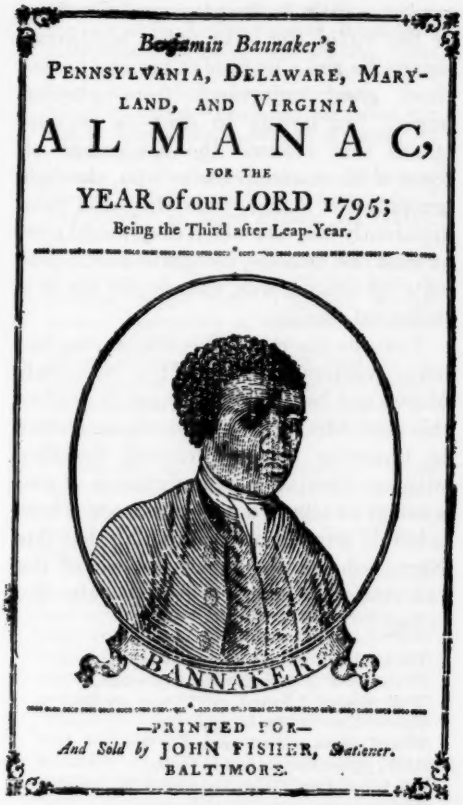
The publishers of the almanac, in their editorial notice, praise it as "having met the approbation of several of the most distinguished astronomers of America, particularly the celebrated Mr. Rittenhouse." (See [3], p. 82.)

Secretary of State Jefferson replied to Banneker in part:

> Nobody wishes more than I do to see such proofs as you exhibit that Nature has given to our black brethren talents equal to those of the other colors of men, and that the appearance of a want of them is owing only to the degraded condition of their existence both in Africa and America. . . . I have taken the liberty of sending your almanac to M. de Condorcet, secretary of the Academy of Sciences at Paris, and member of the Philanthropic Society, because I consider it a document to which your whole color has a right for their justification against the doubts which have been entertained of them. (See [3], p. 83.)

In his letter to Condorcet, Jefferson gave an enthusiastic account of the mathematical and architectural abilities of Banneker. (See [9], p. 118.)

Banneker's almanac of 1793 included his proposal for the establishment of the office of Secretary of Peace in the President's Cabinet and his overly idealistic,



pacifist plan for the insuring of national peace [10].

He continued the publication of a yearly almanac for ten years until declining health forced him to discontinue his calculations.

The final years of the life of this Negro intellectual certainly were in contrast with the way of life of his less-fortunate fellow Negroes and, indeed, of the more-fortunate white men of the time who were, for the most part, burdened by the struggle of seeing a fledgling nation through the turbulence of adolescence. Freed from the necessity of laboriously providing for his physical needs, Banneker was now able to keep the long night vigils under the moon and stars that fascinated him so much. His daylight hours, consequently, were partitioned: a little to the cultivation of his

garden, a little to the playing of the flute or the violin (for, like many mathematicians, he was a lover of music and he derived great enjoyment from playing music), but largely to sleep—a circumstance that aroused the displeasure of some of his contemporaries who, through ignorance or malice, were critical of that apparently lazy life which they would now at least, let us hope, recognize and appreciate for what it was, namely the life of a dedicated scholar.

Towards the end of his life, among his many visitors was one Mrs. Susannah Mason and her daughter. Some days after this visit, Mrs. Mason sent a rhymed letter to Banneker which contained, together with an exhortation to uprightness of life, a eulogy of him as a man of science. There is hardly any reason for thinking that this Negro scholar was much in need of the exhortation. Hence I record only the eulogy:

Transmitted on the wings of fame,
Thine *éclat* sounding with thy name,
Well pleased I heard, ere 't was my lot,
To see thee in thy humble cot,
That Genius smiled upon thy birth,
And application called it forth;
That times and tides thou couldst presage,
And traverse the celestial stage,
Where shining orbs their circles run
In swift rotation round the sun;
. . . . . .
Thou, a man exalted high
Conspicuous in the world's eye,
On record now thy name's enrolled;
And future ages will be told
There lived a man named Banneker,
An African Astronomer!

Banneker's peaceful death occurred on October 9, 1806.

It remains to attempt to estimate the place of Benjamin Banneker in the history of mathematics. It is of course clear that there was nothing in the accomplishments of Banneker to secure for him a position in the history of creative mathematics; the circumstances of the land and the country of his birth account for that. Scholarship rarely flourishes except under the stimulation of an established culture. The history of the first three centuries of what are now the United States is without any notable achievement in mathematics. In the sixteenth and seventeenth centuries the mathematical needs of the Colonies were very few, and the pressing problems confronting the struggling nation left little incentive or opportunity for scholarly interests. The eighteenth century was the first in our land when any special interest was shown in mathematics.

A definitive picture of what mathematics was in America in that century is available in David Eugene Smith and Jekuthiel Ginsburg: *A History of Mathematics in America Before 1900.* The most enjoyable, and possibly the best, way to make the acquaintance of the mathematics of any period in history is to study the lives and writings of the individuals whose mathematical work or influence on mathematics dominated that period. The four individuals—two astronomers and two public figures—whom Smith and Ginsburg single out as outstanding in eighteenth-century mathematics in America are: John Winthrop (1714–1779), scholarly professor of astronomy at Harvard; David Rittenhouse (1732–1796), watch and instrument maker, surveyor, and distinguished astronomer; Benjamin Franklin (1706–1790), who, by his encouragement and support of the study of mathematics, of schools, and of printing, and by the opportunities he opened up for Americans to become acquainted with European scientists and scientific societies, played a significant role in the history of mathematics in America; and Thomas Jefferson (1743–1826), who was an ardent promoter of the study of mathematics by his writings and especially by the influence he exercised in the shaping of the curriculums of the University of Virginia and the United States Military Academy, as also by the encouragement that he won for American scholars by his association with the intellectual leaders of Europe [11].

From a study of the history of mathematics in eighteenth-century America, it is evident how close Benjamin Banneker

came to being representative of the mathematicians of his day. His interests in mathematics and applied mathematics were precisely those that characterize the mathematics of the eighteenth century in America: problem solving, almanac computation, surveying, and astronomy. A notable ability in all of these endeavors, together with his association with Thomas Jefferson and Andrew Ellicott, and the fact that he was the first Negro in the history of our country to win distinction as a student and worker in mathematics, would seem to have secured for Benjamin Banneker an honorable, albeit minor, place in the history of mathematics in America.

The reproduction (on page 35) of the title page of Banneker's almanac of 1795 (note the variant spelling of his name) was made from a photocopy of the original by the Maryland Historical Society and is used with the Society's kind permission.

Notes and References

1. Martha E. Tyson, *Banneker, the Afric-American Astronomer*, edited by her daughter, Anne T. Kirk (Philadelphia: Friends' Book Association, 1884). All the quotations in the article and notes that are not credited to other sources are from this short but excellent monograph.
2. McHenry's reference to *double fractions* occurs in an extremely laudatory letter of commendation of Banneker and his *Pennsylvania, Delaware, Maryland, and Virginia Almanac and Ephemeris*, on August 20, 1791, to the publishers of the almanac ([1], p. 49). The expression *Double Position* is quoted elsewhere ([3], p. 80) in this same connection. There seems to be reference here to the "rule of double false," also called the "rule of false positions." See David Eugene Smith, *History of Mathematics* (Boston: Ginn and Company, 1925), II, pp. 437–439. In Shirley Graham's *Your Most Humble Servant: The Story of Benjamin Banneker* (New York: Julian Messner, Inc., 1949), we have a very pleasant account from the pen of an excellent writer. Banneker's mathematical training and reading, however, were not as extensive as this fictional biography might lead one to believe.
3. M. D. Conway, "Benjamin Banneker, the Negro Astronomer," *Atlantic Monthly*, XI (1863), pp. 79–84.
4. Henry E. Baker, "Benjamin Banneker, the Negro Mathematician and Astronomer," *The Journal of Negro History*, III (1918), pp. 99–118.
5. David Eugene Smith and Jekuthiel Ginsburg, *A History of Mathematics in America Before 1900* (Chicago: Mathematical Association of America and Open Court Publishing Co., 1934).
6. *Scripta Mathematica*, II (1933–1934), p. 73: a summary of the sketch of the New Hampshire almanac maker Dudley Leavitt (1772–1851) in the *Dictionary of American Biography*, Vol. XI.
7. "There was an especial sympathy," wrote Mrs. Tyson, "between one of the younger members of the Ellicott family and Banneker. This was George Ellicott, the father of the author.

   "He had been a mere youth when the family settled at Ellicott's Mills. That he had much energy and business capacity, we may know from the fact that the present road from Frederick to Baltimore was surveyed and laid out by him when he was seventeen years old. . . . [He] had a great love of the science of astronomy in which he was well versed. Conceiving early a high estimate of Banneker's powers, he found pleasure in making him frequent visits, and furnishing him with such books and instruments as would aid him in his studies."
8. Major Andrew Ellicott (1754–1820) served his country on numerous important surveys of state and territorial boundaries. He was professor of mathematics at the United States Military Academy from 1812 until his death, and he exercised considerable influence on American mathematics during this period ([5], p. 84).
9. Adrienne Koch, *The Philosophy of Thomas Jefferson* (New York: Columbia University Press, 1943).
10. The complete peace plan is given in Carter Godwin Woodson, *The Negro in Our History* (Washington, D.C.: The Associated Publishers, 1922), pp. 138–140.
11. David Eugene Smith, "Thomas Jefferson and Mathematics," *Scripta Mathematica*, I (1932–1933), pp. 3–14.

• NEWS NOTES

from the Conference Board of the Mathematical Sciences,
G. Baley Price, Washington, D.C.

The Conference Board of the Mathematical Sciences had its origin in the War Policy Committee,* which was appointed by the American Mathematical Society and the Mathematical Association of America at the end of 1942 to deal with some of their common problems which arose out of World War II. The War Policy Committee was supported by a grant from the Rockefeller Foundation. The war over, the War Policy Committee was discharged in November 1945. The American Mathematical Society immediately took the lead in the formation of the Mathematical Policy Committee, usually known simply as the Policy Committee.† This committee grew and eventually had six of the mathematical organizations as members.

In 1958 the Policy Committee was developed into the Conference Organization of the Mathematical Sciences, and a constitution and by-laws were drawn up. In December 1958, the Mathematical Association of America received a grant from the Carnegie Corporation of New York for establishment of a Washington, D.C. office. At its Salt Lake City meeting in 1959 the Association recommended that the Washington office be established by the Conference Organization with the Carnegie grant. The Conference Organization accepted this responsibility, and on February 25, 1960, the Conference Organization was incorporated in the District of Columbia with the new name, Conference Board of the Mathematical Sciences. G. Baley Price, who had been appointed first Executive Secretary, opened the Washington office July 1, 1960.

The Conference Board of the Mathematical Sciences has six member organizations and no individual members. The six member organizations are the American Mathematical Society, the Association for Symbolic Logic, the Institute of Mathematical Statistics, the Mathematical Association of America, the National Council of Teachers of Mathematics, and the Society for Industrial and Applied Mathematics.

The Washington office will not be involved in any way in the operation of the activities of the member organizations, and it is not expected that any of the activities of the member organizations will be transferred to the Washington office or to the Conference Board.

The Washington office will gather information about events and developments, especially those in Washington, which concern mathematics and relay this information to the member organizations, mathematicians, and others who may wish to receive it.

The Washington office will supply information and help on mathematical matters as the opportunity arises. Further, in carrying out this function, assistance from the member organizations and individual mathematicians will be arranged for as the situation demands and opportunity permits. Member organizations may ask help of the Washington office.

From time to time the Washington office will manage or operate special projects which are compatible with the purposes and functions of the Conference

* The history of the War Policy Committee is sketched in the following: *American Mathematical Monthly*, 50 (1943), 138, 205, 466, 593; 51 (1944), 112–115, 549; 52 (1945), 115. *Bulletin of the American Mathematical Society*, 49 (1943), 199.

† The first reference to the formation of the Policy Committee occurs in the report of The Twenty-Ninth Annual Meeting of the Association, *American Mathematical Monthly*, 53 (1946), 178.

Board and which will serve the common interests of the member organizations. These projects may be supported by contracts or grants from foundations, government agencies, or other organizations.

The management of the Conference Board is vested in its Council, which consists of two representatives from each of the member organizations and six representatives at large. The officers of the Conference Board are the following: Chairman, S. S. Wilks; Secretary, J. R. Mayor; Treasurer, A. E. Meder, Jr.; Executive Secretary, G. B. Price.

Letter to the editor

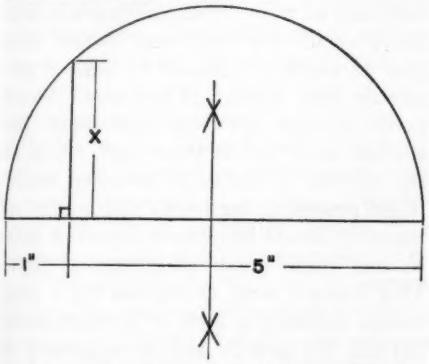
Dear Editor:

I would like to submit the following method for the construction of a radical in geometry.

A standard problem on geometric constructions is: Construct a line segment equal to $\sqrt{5}$ inches. This is merely the mean proportional between two line segments of 1 and 5 inches.

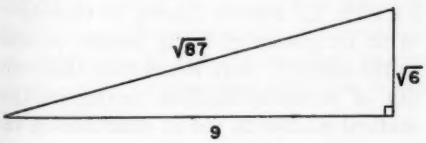
$$\frac{1}{x}=\frac{x}{5}, \qquad x^2=5, \qquad x=\sqrt{5}.$$

The student marks off a line segment of 1 inch and then one of 5 inches and bisects this 6 inch line segment. Construct the semicircle from the midpoint and erect a perpendicular, and we have constructed $x=\sqrt{5}$.

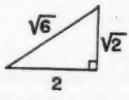


Now, suppose we were asked to construct a segment $\sqrt{87}$ inches long. Would we mark off one segment of 1 inch and then 87 more 1 inch segments and construct the mean proportional? I hope not. I am sure most students would lose count before they have the required 87 inches and the paper needed would be large indeed!

Consider this method: Use $\sqrt{87}$ as the hypotenuse of a right triangle and determine the closest integral square root of 87, which is 9 since $9\cdot 9=81$. Thus 9 will be one of the legs and the Pythagorean Theorem gives the other as $\sqrt{6}$.

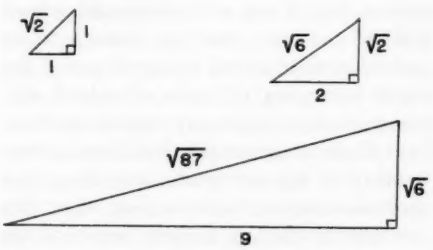


Next, do the same for $\sqrt{6}$. The closest integral square root is 2, since $2\cdot 2$ is 4. We have:

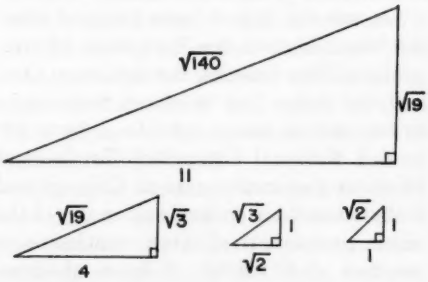


The last triangle gives:

Reverse this analytical process to obtain the synthetic construction:



Let's try one more. $\sqrt{140}$ would give:



The actual construction would consist of making the constructions in reverse order.

I have never seen this construction in any geometry textbook. My students follow the method very well. It seems to tie together radicals, the Pythagorean Theorem, and geometric constructions.

Yours truly,
FREDRIC V. LANE
San Fernando High School
San Fernando, California

• POINTS AND VIEWPOINTS

A column of unofficial comment

As another sees us

by Marcus Riske, Wellington, New Zealand

Editorial Note: Arriving in this country in August 1959 to work as an associate teacher in the University of Illinois School Mathematics Project, Mr. Riske has views on our teaching of interest to all of us.

Ten months in the United States studying, teaching, and observing is small enough qualification to speak with assurance about any aspect of your education system, but I am an experienced school teacher and have traveled enough to be quite conceited about the accuracy of my rapid surveying of parts of school systems. In any case, many Americans have been kind enough to say that there is some validity in my comments, and they show no little interest in them. For what they are worth, then, I present some of the conclusions I arrived at during a most informative stay in the United States.

Let me say that I have traveled from the West Coast to the East, from Minneapolis to New Orleans, through more than half the states and in about three score towns and as many schools. I have attended National Council of Teachers of Mathematics conferences at Chicago and Buffalo and have spoken with many of the most prominent of your mathematics teachers and writers; I have observed many teachers in classes, both in UICSM* work and otherwise. I have read a great deal about your mathematical education. Much of what I have to say is conditioned by my greater acquaintance with UICSM courses than with any others, but the specific criticisms are largely true of the general situation in classrooms and school systems, so far as I am able to judge.

My first impressions have been so convincingly confirmed that I must state them at the very beginning of this discussion. Your mathematics teaching at high school level is seriously handicapped by the year-at-a-time scheduling system which is universal in this country. No scheme I have met at the level of eighth grade and above avoids the traditional notion of a year of algebra, followed by one of geometry, then algebra II and some hotchpotch in the last year (provided the student stays with it that long). Much of the adverse criticism of so-called traditional programs, textbooks and quality of teaching should have been aimed at this, to me, quite crazy allocation of time. Five hours a week of algebra for a year means reaching a level of sophistication beyond the growth and development of the student. Especially true is this when he has had no science at all worth calling such. The mathematics becomes a set of tricks to be performed, however much the course is modern enough to supply justifications in well-enunciated axioms. Whatever you might do, the fact remains that quadratic equations to an eighth or even ninth grader are meaningless without relation to the real world of problem solving found in his science classes, or elsewhere. With no geometry being taught parallel to the algebra, there is not even the benefits of cross-fertilization among mathematical studies to aid in concretizing the

* University of Illinois Committee on School Mathematics.

concepts. Just too much of the same branch of mathematics is taught on end—and then left in some sort of hopeful cold storage for fifteen months!

In the second year the level of understanding required for the new subject is far beneath the student, who should have begun this work a year sooner (two or even three years sooner in some countries). Stage A geometry is largely empirical construction work in most courses. I have seen tenth grade students bored to tears fiddling with stuff that would have entranced them a year and a half earlier.

Just how much you expect students to retain from one June to a year from the following September I do not know. But it is an elementary truism that we largely forget what we do not constantly use; another is that spaced learning over longer periods of time is best. Your system allows a student to take two years of mathematics in, say, eighth and ninth grade and then pick the subject up at college several years later. To anyone brought up in a school system in which part of each subject is taught annually, increasing in difficulty as the student matures and acquires skill, this system seems uneconomical and foredoomed to unnecessary repetition and almost inevitable failure. The atomizing of the total mathematical concept is surely not the best way to engage the youth's interest in the whole notion of a mode of thought. Imagine the advantages of discussing, say, squaring and extraction of the square root in arithmetic, algebra, and geometry in the same week or two. I cannot understand how Americans succeed in teaching without this integration naturally arising from the time-tabling of the varying aspects of the same subject matter during the same school period. Trigonometry arises naturally from geometry and is frequently taught as part of this in the first instance, just as logarithms arise out of discussions of exponents in algebra and are taught much earlier than appears the case in the United States.

So I conclude that one of the most deleterious features of your old, and new, mathematics teaching comes from a fixed notion of subject order and time distribution. I have been repeatedly told, "we are stuck with this," and that I do not realize the mobility of population here. Neither seems of importance compared with the importance of having the student taught at a level of difficulty comparable to his age and in a program that shows interconnection of the various aspects of the one discipline.

Now to my next area of criticism—good points I shall leave to the last to encourage you to complete this diatribe. I find much of the teaching verbal and formal beyond anything I had previously imagined. There seems to be a distinct fear of the student's putting pencil to paper in class. I have always been led to believe that mathematics is something that one *does* as well as talks about. UICSM teachers are guilty of this excessive verbalism but are not at all alone in it. I doubt that any of the newer high school projects lend themselves to understanding by doing, as much as they demand talk, question and answer, talk, as the dominant mode of understanding. Indeed, most of my acquaintance with American students at home and in this country leads me to believe that actual writing of examples is so little practiced as to be a lost art. And I have found it not easy to have my students here write and solve in class. They prefer to have discussions and to write at home, if at all.

Of course, we may be wrong when we say that a good mathematics lesson is one-quarter teaching and three-quarters working examples by students, but certainly 95 per cent discussion in class with scratch work at home is wrong—and this is no exaggeration of what I have seen. Textbooks and "modern" courses, UICSM no exception, tend to this, although teachers are to blame more than books. I can only hope that I have been unlucky in my sampling of teachers!

Formalism excessively written into the new texts might be a needed reaction to so-called cookbook teaching. This, however, is a new danger in itself. Theorem proofs in elementary algebra may well become an end in themselves, as Euclidean proofs were in my day, and still the student is no better equipped to attack equation solving. I must stress the level of child-sophistication as the main factor in the degree of abstract reasoning expected of a class. There is no reason at all why elementary equations may not be successfully and enjoyably solved without the theoretical justification of modern axioms in algebra. These can as well arise from the doing of the equations and the growing perception of what is "behind" the working, and at a later stage of development. UICSM at times consciously uses this simple truism; at others, it burdens the teacher and class with excessive, mere theorem proving.

My last point of criticism is that so much of the teaching is so extremely dull. The teacher and the class go through the motions in such a dead-and-alive manner that I am astonished at the excellent behavior of the students. I must watch many New Zealanders actually teaching mathematics when I return home to see if I am not nostalgically romancing when I think they are livelier than their American colleagues. I do know that dull teaching tends to result in uproar among our students. Here I found docility in excess—or were they being polite to a visitor? I purposely came late to classes and was seldom expected or announced, yet always, without fail, the kids were quiet, and so few participated or appeared vitally interested. Exceptions there were, of course, but generally I have found corroboration of my estimate from young people at high school or within a few years of leaving high school. No new schemes of work can succeed against this lack of fire in teachers.

Almost all the experimental programs I have seen and examined are for the abler pupils. This is natural enough—this is where the need for trained minds will find its satisfaction. Is there any consistent work being attempted to give the next level of intelligence mathematical training in line with its special needs? This is a world problem that has not been solved with either the watered-down programs of the same work given the bright student or the so-called general or core mathematics. Perhaps it is a country like New Zealand, with its smaller amount of industrial automation and its perpetuation of the craftsman through wide apprenticeship schemes, that must pioneer mathematics for boys, anyway, suitable to their interests and future needs.

Comments in praise of aspects of mathematics education in the United States, as I have seen it, come next. The varying programs which aspire to infusing some of the newer aspects of mathematical content and thinking into high school courses can call forth nothing but praise from any person interested in the future of his own country. The work pioneered in the United States deserves and gets the thanks of the rest of the world. Long overdue are these refurbishings of mathematical textbooks and thinking at high school level. No country hoping to train a new generation to fulfill the needs of the next generation can afford to neglect this outstanding pioneering work. My personal debt to UICSM, SMSG, Maryland, Madison, Ball State, can only be paid by trying to infuse something of this spirit into my own country, where we excel at manipulative skills far beyond the comparable age and intelligence of students here, but where we neglect until far too late the rationale behind the manipulation.

The emphasis on exploratory exercises leading to discovery of principles is another excellent aspect of newer programs. Nowhere is this better done than in some parts of the UICSM courses. The Socratic method is the oldest of the good teaching methods and is most timely in its

revival in mathematics. That this is making some teachers afraid to teach or to explain at length when students are stuck cannot be helped. Every good method in educational history has been fouled up by those who misinterpret it.

Another well-timed improvement is precision of language in mathematical texts. Even if this becomes irksome to the older teacher, this emphasis, even overemphasis, is along right lines today. Sloppiness of linguistics in mathematics and science has been, and probably still is, the mode. (Will it ever be otherwise in politics, religion, social affairs, educational theory?) But it is really unforgivable. Verbal precision and logical consistency are the worthy aims of many of the newer courses and, to a great degree, success has been achieved.

Experiments in actual content seem less striking, as far as I can judge. Axioms to justify manipulation are, I would say, less a change in content than a change in logical approach. The most noteworthy content change might well come lower down in the school system than in the high schools, in acquaintance with number systems rather than concentration on our overfamiliar counting numbers. Some really new work in geometry is being tentatively tried by UICSM, a most interesting beginning that might become a truly novel breakthrough in a year or so. Ball State has worked somewhere off traditional Euclidean lines, but not startlingly. Perhaps when the grade school experimentation of such men as David Page and Robert Davis reaches fruition, real content changes from eighth grade up will be realizable. Certainly the changing emphases from arithmetical skills (which occupy so much of our English time because of the craziest of all money systems!) and mere memorized skills, to the understanding of the mathematical systems behind all our practical arithmetic are, in my own opinion, along the best of lines.

It will take some time for all your experimental systems—no, that is too strong a word; let us say projects—to fuse into books available to the ordinary, conscientious classroom teacher and forward-looking administrator. When the day comes that it is not Begle OR Beberman, but the results of the union (or is it "intersection"?) of the two sets, as it were, the world of mathematics teachers will have much to thank the United States for. May I be early in the field and thank the great many people who have contributed so much to a most profitable year in the United States?

New uses of mathematics

The biologist is applying mathematical theory to the study of inheritance; industry is using mathematics in scheduling production and distribution; the social scientist is using ideas from modern statistics; the psychologist is using mathematics of game theory. In fact, the logic of mathematical models shows promise as the basis for developing teaching machines for all areas of knowledge. The new uses of mathematics require less manipulation of formulas and equations but greater understanding of the structure of mathematics and mathematical systems. There is less emphasis on human computation that can be done by machines, and more emphasis on the construction of mathematical models and symbolic representation of ideas and relationships. Because of these new uses, mathematics is being firmly woven into the fabric of the national culture. The role of mathematics is not only to grind out answers to engineering problems, but to produce mathematical models (prototypes) that forecast the outcome of social trends and even the behavioral changes of the group. Such important new uses and interpretations of mathematics require that students have a program with a greater depth than the classical program designed for nineteenth century education. The demands of society require a thorough revision of our present secondary school mathematics curriculum. —*From the* U.S. Office of Education Bulletin OE-29016, *"Improved Programs in Mathematics Require Inservice Education for Teachers."*

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Algebra and Trigonometry, Edward A. Cameron (New York: Henry Holt and Company, 1960). Cloth, xi+290 pp., $5.90.

To evaluate this book and its suggested use in high school or college, it is necessary to describe the content so far as it is indicative of the attempt to blend the old with the new and to "prepare the student for analytic geometry and calculus," the purpose stated on the jacket.

The emphasis in Chapter I is on mathematical structure. In systematic form, the properties of the real number system are stated and the justification for "manipulative maneuvers of elementary algebra" (p. 4) is given. Rigorous proofs of theorems follow, and a good set of exercises.

Chapter II introduces sets, its language and symbolism, and Venn diagrams. The function concept is first introduced in conventional manner, then defined in terms of sets. Great care is taken to clarify the concepts of domain and range and to distinguish between functional and nonfunctional relations.

Chapter III is on equations, Chapter IV on inequalities. The emphasis on equivalent systems, the work on absolute values, the graphs of inequalities in one variable, and the derivation of the quadratic formula are all strong points. I feel, however, that "Order Properties of the Real Numbers" (p. 38) should follow the equality, addition and multiplication properties for real numbers on pages 3 and 4. More weight should be given to associating the real numbers with their graphs on the number line, or ordered number pairs with points in the plane. These topics were merely mentioned at the end of the first and second chapters, respectively. The relation: if $a<b$, the graph of a is to the left of the graph of b, for all real numbers a, b, etc., should be firmly established. The property $a+c<b+c$ will follow easily. I think the spirit of "modern approach" and, above all, the appreciation for the behavior of real numbers would be better served if equations and inequalities were treated simultaneously, particularly since the presentation and vocabulary used would lend themselves to this change. Open sentence, replacement for the variable x, solution set, scope of variable, equivalent sets, all indicate the effort of the author to acquaint the reader with these ideas.

A traditional treatment of systems of linear equations is given in Chapter V. It seems too bad that topics are so strictly divided and that open sentences in two variables expressing inequalities or systems of inequalities are not even mentioned. For that matter, after the chapter on inequalities, no further application is made.

Chapter VI, on matrices and their determinants, is very valuable. Concepts of minors and cofactors are introduced, and theorems are derived that lead to Cramer's Rule. Properties of matrices are presented and the student is exposed, even though briefly, to an algebraic system in which multiplication is not commutative. (On p. 76, I do not appreciate the fabrication of the word "generalizable"; a proper verbal form could well replace it.)

Chapters VII and VIII are concerned mostly with topics of an intermediate algebra course. Care is taken in each section, however, to define the domain carefully in order to make statements valid, and proofs are included in the exercises.

In Chapters IX and X, on trigonometry, diagrams representing positive and negative angles of any magnitude and the definition of the standard position of the angle lead naturally to definitions of the six trigonometric functions for any angle. Special cases follow. The emphasis is on the function concept. The circular function is carefully defined and distinguished from the trigonometric function, and then again related to it. The formula for the cos $(A-B)$ is derived from the distance formula (derived in Chapter II) and serves in turn as the basis for other formulas. Much of this section is in accordance with the Report of the Commission on Mathematics (see Appendices, CEEB New York 1959, pp. 202, 203) and has its strong points.

Chapter XI, on complex numbers, again places the emphasis on structure. The system is defined as the set of all ordered pairs (a, b) of real numbers, and it is shown that the properties and theorems derived in Chapter I hold. The concept of isomorphism is introduced. It is a good chapter. (I find it difficult to accept the phrase "the dummy symbol i," on p. 228.)

Chapter XII presents the Remainder and Factor Theorems, Synthetic Division, theorems on polynomials and polynomial equations. All are important but are isolated from the rest of the book. I should like to suggest the treatment of polynomials as given in the First Course in Algebra of the School Mathematics Study Group. This type of material, beautifully presented, would enrich Chapter XII and unify it with Chapters I–V.

The foregoing description is an attempt to show to what extent *Algebra and Trigonometry* offers approaches and materials different from those of the conventional textbook. I believe it could be used to good advantage with eleventh

or twelfth grade students, as a first exposure to some topics and possibly a review of others. It might be particularly suited to schools which have not adopted programs of SMSG caliber.—*Martha Zelinka, Weston High School, Weston Massachusetts.*

Instruction in Arithmetic, Twenty-fifth Yearbook of The National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1960). Cloth, viii+366 pp., $4.50 ($3.50 to Council members).

The stated purpose of the Twenty-fifth Yearbook is " . . . to serve the needs of teachers of arithmetic, principals and supervisors who work in the elementary school, and students in elementary education." Although I am not an authority on elementary-school education or the teaching of arithmetic, I feel qualified to say that the Twenty-fifth Yearbook definitely meets its stated objective and is a most valuable reference for anyone concerned with the teaching of arithmetic. However, I suspect that many members of The National Council of Teachers of Mathematics are asking the question, "Is the Twenty-fifth Yearbook of value to the teacher of high school mathematics?" To answer this question, we must examine the structure and content of the yearbook.

The first chapter is an introduction written by Foster E. Grossnickle, chairman of the yearbook committee. This chapter is a very adequate review of the entire book, and I will make reference to many of the ideas it contains. Dr. Grossnickle points out that the book may be divided into five parts. He states, "Part one deals with the cultural value of arithmetic, its structure, learnings peculiar to this subject, and its content. Chapters 2–5 constitute this segment of the book. Part two deals with the learner and factors affecting him in learning arithmetic. Chapters 6–10 comprise this phase of the subject. Part three deals predominantly with the meaning of modern mathematics and its impact on arithmetic. Chapters 11 and 12 deal with this area. Parts four and five consist of one chapter each. Part four deals with the training of teachers of arithmetic. Part five gives a brief summary of investigations and other publications dealing with arithmetic since the presentation of the last yearbook in arithmetic by the Council in 1941."

The first chapter of Part one presents a brief cultural history of arithmetic. It covers such topics as the history and make-up of numeration systems, impact of arithmetic on various cultures, use and importance of arithmetic in daily life, and the importance of arithmetic in the future. Although many interesting and important ideas are presented in the chapter, the writer was hampered by the necessity of covering too much material in too short a space. As a result, a few of the ideas presented are not developed enough to make them valuable to the reader.

The next topic is titled "Structuring Arithmetic." The authors compare the word "structure" as used in arithmetic to the word "structure" as used in the construction of buildings and present the idea that if a student can see how arithmetic is "constructed," he will be better able to understand and use the subject. The chapter points out ways in which structures of arithmetic concepts are developed from physical experiences. It stresses the importance of getting a close fit between a physical experience and the mathematical idea to be developed from it. Noteworthy examples are given that show how various visual materials and devices can be used to structure our decimal numeration system and such basic ideas as ordering, and the commutative, associative, and distributive principles. This is followed by a discussion of how to structure the concepts of rate pairs, fractions, and per cents.

Basic generalizations that students should make in arithmetic, business applications of arithmetic, and elementary geometry are listed in Chapter 4. A considerable number of exercises are given that should enable the teacher to help the pupil make important generalizations for himself. This chapter not only provides a wealth of ideas for teachers of elementary-school arithmetic, but it also contains many ideas that could be applied in higher grades.

A chapter on "Arithmetic in Kindergarten and Grades 1 and 2" is the only one in the book dealing with curriculum. The author makes it clear that there is no clear-cut program of arithmetic instruction for kindergarten and grades 1 and 2. He presents some of the basic issues that are involved in the teaching of arithmetic at these grade levels and makes suggestions for a unified program of arithmetic for the three grades.

The second group of chapters outlined by Dr. Grossnickle deals with factors that directly affect pupils in their study of arithmetic. It starts with a chapter that outlines the problems of dealing with individual differences in the arithmetic classroom. Differentiation of instructional materials, acceleration of pupils, and homogeneous grouping are discussed as means of alleviating such differences.

This is followed by a chapter that emphasizes the importance of an adequate guidance plan in building a strong arithmetic program. It includes material on adjustments that can be made for the slow, fast, and average learner.

The relationship between the mental attitude of pupils and their success in arithmetic is another topic covered in this section of the yearbook. Although the chapter is mainly devoted to general principles of good mental health, it does include some specific ideas relating to arithmetic.

As every mathematics teacher knows, reading and mathematics are closely related. Therefore, it is quite appropriate for the yearbook to contain a chapter on "Reading in Arithmetic." The first part of the chapter deals with sources

of difficulty in mathematical reading, but makes few constructive suggestions on how to deal with the difficulties listed. The second part presents a review of recent research dealing with the relationship between reading and problem solving, and suggests ways to help pupils develop skill in this area.

This section ends with a chapter that gives a review of many types of instructional materials (including textbooks), suggests criteria for the selection of such materials, and lists some common and useful supplementary materials that can be used to develop a wide range of mathematical ideas.

The first four chapters of this second group of chapters are definitely pointed toward the elementary school. They contain a considerable amount of material that is often found in education magazines and educational psychology textbooks. However, the fifth chapter contains some ideas that would prove valuable to many teachers in grades 7 to 12.

The third group of chapters deals with modern mathematics and its relationship to arithmetic. It contains a very valuable chapter that lists and attempts to clarify and standardize many of the important definitions of arithmetic. Although I am certain we shall never see the day when this task is accomplished, this is an important step in the right direction.

A chapter that discusses the importance of modern mathematics in an adequate arithmetic program is also presented. The chapter covers the set concept and discusses the use of number scales and symbols. This is a valuable chapter for any mathematics teacher.

The book also contains a chapter that deals with the mathematical training of elementary teachers and closes with a selected, annotated bibliography that alone is worth the price of the book.

Each chapter has been well written by men and women who are authorities on their given topics. The Yearbook Committee of the National Council is again to be complimented for an excellent piece of work.

If you are a high school mathematics teacher, don't expect this yearbook to be as valuable a reference as the two that preceded it. Nevertheless, the high school mathematics teacher should have a knowledge of the previous mathematics development of his students, and thus should be familiar with the practices and ideas used in arithmetic teaching. Hence the book is a worthwhile addition to your reference shelf.—*Thomas E. Yager, John Burroughs School, St. Louis, Missouri.*

INSTRUCTIONAL MATERIALS

Formulas in Mathematics, International Film Bureau, Inc., Chicago, Illinois. 10¼ minutes running time, color, $110.

This film was designed to show students how a formula can be used in solving a problem. Actually, it is best suited for reinforcing the concept of like units in translating a problem from English to mathematics and then in using the mathematical answer. It can be used best in grades 7 and 8.

A short, 90 m.p.h. trip in a small, propeller-driven plane furnishes the story for the film. The take-off is followed by several close-ups of the plane and a look at the fields below. The development of the formula $D = r \times t$ includes the ruling out of data, such as fuel consumption and altitude, which are available but not appropriate to the problem at hand. A landmark (river) helps the pilot realize that he must increase his speed. The situation forces the re-use of the distance formula. A contrast of results obtained from using 45 as the number of minutes or ¾ as the number of hours points out the importance of consistency in units. Next in sequence are other applications of the formula $D = r \times t$. The film ends with a dramatic shot of the landing.

It is the airplane flight, not the mathematics, which is dramatized. The color and photography are excellent and add quality to the story. The sound of the airplane is much more effective than the sound of the commentator's voice. Greater color and enthusiasm in the voice might have made it possible for the mathematical content to compete for attention with the drama of the flight.

The best section is the development of the concept of comparable units of measure. A pattern similar to that which follows is used to draw attention to like units:

$$\underset{\text{miles}}{D} = \underset{\text{miles per hour}}{r} \times \underset{\text{hours}}{t}$$

Color is used effectively to point attention to the repeated terms "miles" and "hours." Then a number of applications of the formula to other situations are illustrated to reinforce the idea. Included are the movement of a glacier in feet per year, the laying of a pipeline in feet per hour, and the flight of a missile in miles per minute. It should be noted that the assumption of constant speed is less tenable in the flight of the missile than in the other illustrations. A fourth illustration of the formula involving Indian trips on horseback does not add to the understanding and should probably have been omitted.

The language is quite precise. Where other films contain "dividing miles by hours," this one divides "the number of miles by the number of hours." However, although few of us would object to "the formula helps him answer a question," some might question "formulas are tools." The stated purpose of this film is to give the meaning "problem solving tool" to the concept "formula." It is my belief, however, that the major value of *Formulas in Mathematics* will be as an aid in focusing the attention of students on units of measure and the consequences of comparing noncomparable units.—*Stuart C. Rankin, Detroit Public Schools, Detroit, Michigan.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Stalking solid geometry with knife and clay

by Jack Price, Denby High School, Detroit, Michigan

If the most difficult task for the student in the study of solid geometry is visualization and the most difficult unit is the work with the sphere and its sections, then the combination of the two makes an almost overwhelming problem. Any teacher who has struggled his way through this difficulty even with a hard-working group of students can attest to this.

A solution to this problem was presented quite by chance in, of all places, an adult evening class. One member of the class (by day a machinist, by night a struggling student) sat down at his desk and with a pleased look on his face deposited on the top of the desk a baseball-sized ball of clay and a sharp pen knife. "I've got this unit licked," he said. And he did.

While working on his problems at home he had spied his son's clay and was struck by its possibilities. By rolling the clay into a ball and then cutting it strategically with the knife he could create any section of a sphere necessary for the problem at hand. The clay could then be rerolled, ready for the next problem. Visualization for him was indeed licked.

With this student's blessing, I have adopted this aid for teaching this section in particular and other units where useful. With a sharp knife, the handy-sized ball can be cut quickly and easily with one or two strokes. For example, in cutting off a spherical segment or a zone, only one quick slice is necessary; for a zone of two bases, two slices. A spherical cone can be made easily by pushing the knife to the center of the ball at a small angle to the vertical axis. A twisting motion will pop out the cone. A slice can easily separate it into a spherical segment and a right circular cone. Other parts are made as easily.

The use of clay is not limited, however, to the unit on spheres. In cones, for example, the knife can represent planes passing through the cone at various angles and thus the conic sections can be produced. In the same manner, truncated prisms and the effects of passing planes through various other solids can be shown.

Another interesting use of clay and knife is to produce composite figures. Often the student mentioned earlier would construct in clay the composite figure called for in a specific problem. Then with his knife he would slice it into its components. In this manner he could more readily see the various parts for which he had to find area or volume.

It is not the purpose of this article to advocate that every student in the class carry a ball of clay and a knife. But these can be used by the teacher as a class demonstration or by the student as he sits at home head-scratching over his homework. The use of clay and knife is not the final word, or even a very compelling one, but it is an aid in overcoming at least two great stumbling blocks in solid geometry.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Nominating Committee

The Committee on Nominations and Elections presents its nominees for offices to be filled in the 1961 election. The term of office for the two vice-presidents is two years. Three directors are to be elected for terms of three years.

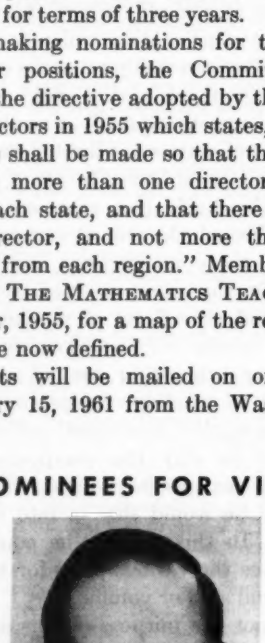
In making nominations for the three director positions, the Committee followed the directive adopted by the Board of Directors in 1955 which states, "Nominations shall be made so that there shall be not more than one director elected from each state, and that there shall be one director, and not more than two, elected from each region." Members may consult THE MATHEMATICS TEACHER for October, 1955, for a map of the regions as they are now defined.

Ballots will be mailed on or before February 15, 1961 from the Washington Office to members of record as of that date. Ballots returned and postmarked not later than March 15, 1961 will be counted.

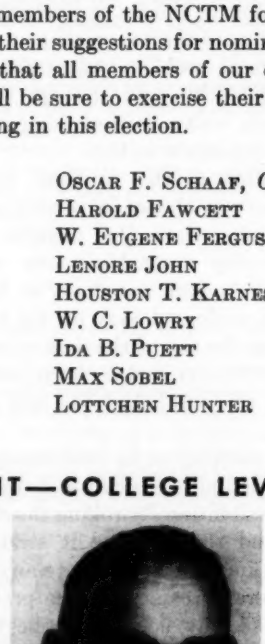
The Committee wishes to thank the many members of the NCTM for help in giving their suggestions for nominees. It is hoped that all members of our organization will be sure to exercise their privilege of voting in this election.

OSCAR F. SCHAAF, *Chairman*
HAROLD FAWCETT
W. EUGENE FERGUSON
LENORE JOHN
HOUSTON T. KARNES
W. C. LOWRY
IDA B. PUETT
MAX SOBEL
LOTTCHEN HUNTER

NOMINEES FOR VICE-PRESIDENT—COLLEGE LEVEL



BRUCE E. MESERVE



MYRON F. ROSSKOPF

Bruce E. Meserve

Professor of Mathematics and Chairman, Department of Mathematics, Montclair State College, Upper Montclair, New Jersey.

A.B., Bates College; A.M. and Ph.D., Duke University.

Teacher of mathematics, Moses Brown School, 1938–41; graduate work at Duke University, 1941–42, 1945–46; Civilian Public Service, 1942–43; U.S. Army, 1943–1945; instructor (1946–48), assistant professor (1948–54) of mathematics, University of Illinois; associate professor (1954–57), professor and chairman of mathematics (1957———), Montclair State College. Teacher in Montclair State College Demonstration High School, 1954–57, 1958–59, 1960.

Fellow, AAAS. Member: AMS, MAA, AMTNE, AMTNYC, CASMT, NJAS, NJEA, Phi Beta Kappa, Sigma Xi, and Kappa Mu Epsilon. Vice president of AMTNJ, 1960———; vice president, Illinois Council of Teachers of Mathematics, 1952–54; chairman, New Jersey Section of MAA, 1957–58. Member: UICSM, 1951–54; NAS/NRC Film Evaluation Board, 1957; Panel on Teacher Training Materials of SMSG; Panel on Teacher Training of the Committee on the Undergraduate Program in Mathematics.

Listed in: *American Men of Science, Leaders in American Science, Who's Who in American Education, Who's Who in the East, World Directory of Mathematicians.*

Activities in NCTM: Member, Board of Directors, 1958–61; Budget Committee, 1959–62; Yearbook Planning Committee, 1957–61. Representative of NCTM on AAAS Co-operative Committee on the Teaching of Science and Mathematics, 1958–62. Past chairman of Committee on Research, Committee on Co-ordination of Mathematics with Business and Industry, Committee on Secondary School Standards. Past member of Secondary School Curriculum Committee, Committee on Institutes and Summer Workshops, and editorial committee of *23rd Yearbook.*

Publications: Coauthor with Virgil S. Mallory and Kenneth C. Skeen of *First Course in Geometry; General Mathematics* (2nd ed.); *First Course in Algebra,* and *Second Course in Algebra.* Coauthor with J. B. Rosenbach, E. A. Whitman, and P. M. Whitman of *College Algebra* (4th ed.); *Essentials of College Algebra* (2nd ed.), and *Intermediate Algebra for Colleges* (2nd. ed.). Author of *Fundamental Concepts of Algebra; Fundamental Concepts of Geometry,* and over forty articles and reviews in professional journals.

Myron F. Rosskopf

Professor of Mathematics, Teachers College, Columbia University, New York, New York.

A. B., M. A., University of Minnesota; Ph.D., Brown University.

Teacher and head, department of mathematics, John Burroughs School, Clayton, Missouri. Associate professor of mathematics and education, Syracuse University.

Member: NCTM, MAA, AMS, Sigma Xi, Phi Beta Kappa, Phi Delta Kappa, Pi Mu Epsilon, AAUP, New York Academy of Sciences.

Activities in NCTM: Associate Editor, THE MATHEMATICS TEACHER; past member and chairman, Yearbook Planning Committee; member, Nominations Committee, 1958; delegate to International Congress of Mathematicians, Amsterdam, The Netherlands, 1954.

Other Activities: Charter member of Association of Mathematics Teachers of New York State; past president of AMTNYS; past member of Council and Executive Committee of AMTNYS; member of writing groups of Commission on Mathematics, CEEB.

Publications: Numerous articles appearing in THE MATHEMATICS TEACHER, *School Science and Mathematics, Teachers College Record,* and other periodicals; chapters in various NCTM yearbooks; coauthor of textbooks on elementary, secondary, and college levels.

NOMINEES FOR VICE-PRESIDENT— JUNIOR HIGH SCHOOL LEVEL

HELEN L. GARSTENS

MILDRED KEIFFER

Helen L. Garstens

Associate Director, University of Maryland Mathematics Project, University of Maryland; lecturer in mathematics and in education, University of Maryland.

B.A., Hunter College, New York City; graduate study at Columbia University, University of Virginia, Georgetown U.

Senior high school teacher in New York City; teacher in elementary, junior high, senior high schools of Arlington, Virginia; supervisor of secondary mathematics, Arlington, Virginia; demonstration teacher for National Science Foundation Summer Institute for Junior High School Mathematics Teachers at University of Maryland.

Member: NCTM, Arlington Chapter of NCTM, AAAS, Washington Academy of Sciences, Pi Mu Epsilon, Phi Beta Kappa.

Activities in NCTM: member; participant in NCTM convention programs; organizer of Arlington Chapter of NCTM; committee member for Washington, D.C. meeting.

Other Activities: Extensive participation as consultant and lecturer in in-service activities for mathematics teachers sponsored by school systems, universities, individual chapters of NCTM; guest lecturer at summer institutes and workshops; Committee for Gifted in Mathematics, NEA; writing of curriculum materials for UMMaP; writing team for grades 4, 5, 6 for SMSG; citation by Washington Academy of Sciences for work in mathematics curriculum development and teacher training.

Publications: Articles in *The Arithmetic Teacher*, THE MATHEMATICS TEACHER; "Experimental Mathematics for the Seventh Grade," *NEA Journal;* contributor to *Mathematics for the Junior High School, First Book, Second Book,* UMMaP.

Mildred Keiffer

Supervisor of Mathematics, Secondary Schools, Cincinnati, Ohio.

A.B., University of Cincinnati; M.A., Columbia Teachers College; workshops at Duke, Rutgers, University of Wisconsin, University of Maryland.

Teacher in elementary, junior, and senior high schools of Cincinnati; supervising teacher, assistant supervisor, supervisor of secondary-school mathematics of the Cincinnati Public Schools.

Member: NCTM, MAA, AAAS, ASCD, CASMT; Ohio Council of Teachers of Mathematics; Mathematics Club of Greater Cincinnati; Phi Beta Kappa; Delta Kappa Gamma; Kappa Delta Pi; OEA; life member of NEA.

Activities in NCTM: Member at var-

ious times of NCTM committees on Supplementary Publications, Membership, Nomination, Television; Editorial Board, THE MATHEMATICS TEACHER; Subcommittee of the Secondary School Curriculum Committee; speaker at various meetings; chairman NCTM Regional Orientation Conference, Cincinnati.

Other activities: Consultant for OCTM Workshops, Shell Program at Cornell; former president of Cincinnati Mathematics Club, Ohio Council of Teachers of Mathematics; SMSG writing teams at the University of Michigan and Stanford University; SMSG Panel for Non-College-Bound Students; Engineering Society Mathematics Tournament Committee.

Publications: Coauthor with Margaret Joseph of *Basic General Mathematics;* contributor to NCTM bulletins and pamphlets, and to *Bulletin of National Association of Secondary School Principals.*

NOMINEES FOR THE BOARD OF DIRECTORS

Max Beberman

Central Region

Professor of Education, University of Illinois; Teacher of Mathematics, University of Illinois High School; Director, University of Illinois Committee on School Mathematics [UICSM] Project.

B.S., College of the City of New York; A.M. and Ed. D., Columbia University.

Teacher of mathematics in Nome (Alaska) public school; Teachers College, Riverdale (New York) Country Day School; associate professor of education, Florida State University.

Member: NCTM, MAA, AMS, CASMT, AAAS, Phi Delta Kappa, Kappa Delta Pi, Pi Mu Epsilon.

Activities in NCTM: Editor, *The Mathematics Student Journal,* 1956–58; member of Publications Board.

Publications: various articles in THE MATHEMATICS TEACHER, *The School Review, Educational Leadership;* author of *An Emerging Program in School Mathematics* (Inglis Lecture); coauthor of *Algebra, Course I; Algebra, Course II; High School Mathematics* [Units 1, 2, 3, 4, 5, 6].

Lurnice Begnaud

Southwestern Region

Supervising Teacher, Southwestern Louisiana Institute, Lafayette, Louisiana; Head of Mathematics Department, Lafayette Senior High School, Lafayette, Louisiana.

B.S., Southwestern Louisiana Institute; M.A., Louisiana State University.

Mathematics teacher, Milton High School, Milton, Louisiana; Carencro High School, Carencro, Louisiana; staff member

MAX BEBERMAN

LURNICE BEGNAUD

JOHN A. BROWN

of Louisiana State University Summer Mathematics Institute, 1950–59; staff member of Duke University Summer Mathematics Institute, 1952.

Member: NCTM, Phi Kappa Phi, Kappa Mu Epsilon, NEA, Louisiana Teachers Association, Association for Student Teaching, American Association of University Women.

Activities in NCTM: state representative for NCTM in Louisiana; past president, Louisiana-Mississippi Section, NCTM; convention program appearance.

Other Activities: Consultant in numerous summer parish (county) workshops throughout Louisiana; participant in convention programs on a local, state, and national level; past president, Mathematics Section of Louisiana Teachers Association.

John A. Brown

Northeastern Region

Professor of Education and Mathematics, University of Delaware, Newark, Delaware.

B.S., LaCrosse State College; M.S., Ph.D., University of Wisconsin.

Teacher at Mattoon High School, Mattoon, Wisconsin; Oconto High School, Oconto, Wisconsin; Merrill High School, Merrill, Wisconsin; Wisconsin High School, University of Wisconsin; State University Teachers College, Oneonta, New York.

Member: NCTM, AAAS, AERA, NEA, MAA Delaware Mathematics Council.

Activities in NCTM: former departmental coeditor; member of Committee on Guidance Pamphlet.

Other activities: member of SMSG junior high panel; SMSG writing team, 1958 and 1959; director of SMSG Center at the University of Delaware; departmental coeditor for *The American Mathematical Monthly;* regional representative of MAA Visiting Lecture Program.

Publications: Contributor to NCTM *22nd Yearbook;* coauthor of textbooks; various magazine articles.

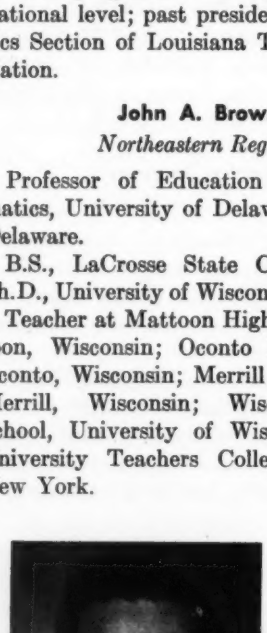
William T. Guy, Jr.

Southwestern Region

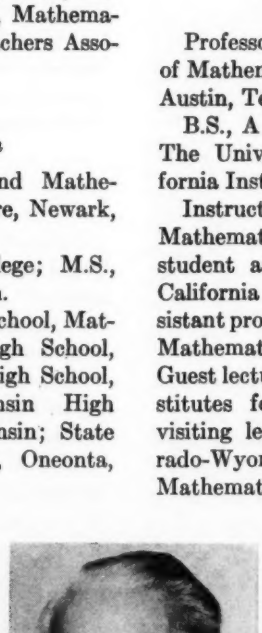
Professor and Chairman, Department of Mathematics, The University of Texas, Austin, Texas.

B.S., A & M College of Texas; M.A., The University of Texas; Ph.D., California Institute of Technology.

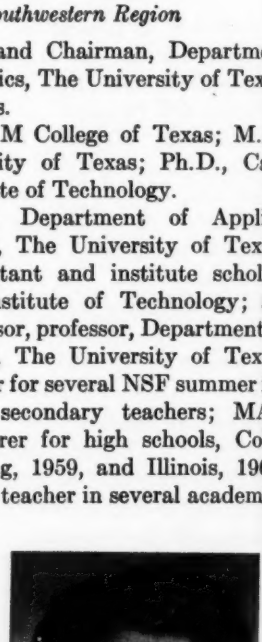
Instructor, Department of Applied Mathematics, The University of Texas; student assistant and institute scholar, California Institute of Technology; assistant professor, professor, Department of Mathematics, The University of Texas. Guest lecturer for several NSF summer institutes for secondary teachers; MAA visiting lecturer for high schools, Colorado-Wyoming, 1959, and Illinois, 1960. Mathematics teacher in several academic-



WILLIAM T. GUY, JR.



JULIUS H. HLAVATY



RACHEL P. KENISTON

year and summer institutes held at The University of Texas; director, 1960 Summer Institute for Elementary School Personnel; director, 1959–60 Visiting Lecturer Program of Mathematics for the Texas Academy of Science; member of Commission on Mathematics, Texas Education Agency.

Member: NCTM, MAA, AMS, Texas CTM; Fellow, Texas Academy of Science; Fellow, AAAS; Sigma Xi, Tau Beta Pi, Sigma Pi Sigma.

Activities in NCTM: several invitational addresses to national and regional meetings; refereed many papers for THE MATHEMATICS TEACHER.

Other activities: governor, Mathematical Association of America; past chairman, Texas Section, MAA; consultant to NSF-sponsored mathematics teacher institutes; member, MAA Committee on Undergraduate Program in Mathematics, Committee on Summer Institutes; AAAS representative at two TEPS and NASDTEC conferences; member, SMSG panel and summer writing project in mathematics for elementary school; consulting mathematics editor for textbooks.

Listed in: *American Men of Science*, *Leaders in American Science*, and *Who's Who in Texas*.

Publications: Contributor to *Mathematics Monthly*, *Bulletin of the American Mathematical Society*, *Journal of the Texas Academy of Science*, Proceedings of the Sixth Midwestern Conference on Fluid Mechanics.

Julius H. Hlavaty

Northeastern Region

Chairman, Department of Mathematics, DeWitt Clinton High School; Instructor, Teachers College, Columbia University.

B.S., City College of New York; Ph.D., Columbia University.

First chairman, Department of Mathematics, Bronx High School of Science, 1938–53; National Science Foundation and New York State In-Service Institutes for teachers of mathematics; fellow, Rocky Mountain Workshop, Commission on Secondary School Curriculum, summer 1938; staff, Curriculum Workshop, Syracuse University, summers 1939, 1940; teacher, Evander Childs High School, New York City, 1930–38, Theodore Roosevelt High School, N. Y. C., 1929–30.

Member: Phi Beta Kappa, NCTM, MAA, Central Association of Teachers of Science and Mathematics; Mathematics Chairmen's Association, New York City; Association of Teachers of Mathematics, New York City.

Activities in NCTM: Chairman, Committee on Mathematics for the Talented (1958———); speaker and panel chairman at national conventions; member, Conference on Directions, 1959; chairman, Planning Committee for Yearbook for the Talented.

Other Activities: Director, Mathematics Commission Program, College Entrance Examination Board; School Mathematics Study Group—advisory committee, executive committee, monograph panel, editorial board, chairman of Committee to Evaluate Sample Textbooks, chairman of Committee on the Gifted; NEA Invitational Conference on the Academically Talented, Washington, D.C., 1958; consultant, University of Illinois Committee on School Mathematics; mathematics committee, School and College Study of Admission with Advanced Standing, 1952–55; test consultant, War Department AGO, 1944–46; mathematics committee, Commission on Secondary School Curriculum (the 8-year study); chairman of Mathematics Sub-Committee of the NEA Project on the Academically Talented Pupil; past president, Mathematics Chairmen's Association and Association of Teachers of Mathematics, New York City; past chairman, Standing Committee on Mathematics, Board of Education, New York City; speaker at numerous in-service training and summer institutes (NSF); consultant for school systems at Yardley, Pennsylvania, Ridgewood and Hacken-

sack, New Jersey, New Rochelle, New York, Lexington, Massachusetts, Los Angeles and San Diego, California; Advisory Conference on Co-ordination of Curriculum Studies (NSF), 1960; Advisory Committee for Educational Policies Commission (NEA), 1960–63; consultant, Metropolitan School Study Council.

Publications: Contributor to: THE MATHEMATICS TEACHER, *College Board Review;* translator, Felix, *The Modern Aspect of Mathematics;* editor, *Mathematics for the Academically Talented Student;* contributor, *Program Provisions for the Mathematically Gifted Student;* author, *Review Digest in Solid Geometry.*

Rachel P. Keniston

Western Region

Teacher of Mathematics and Chairman of the Mathematics Department, A. A. Stagg Senior High School, Stockton, California.

A.B., Smith College, Northampton, Massachusetts; M.A., College of Pacific, Stockton, California.

Positions held: Mathematics teacher in Contoocook, New Hampshire, Enfield, Connecticut, Castilleja, Palo Alto, California, and currently in Stockton Unified School District, Stockton, California.

Member: NCTM, NEA, California Mathematics Council, New England Association for Teachers of Mathematics, California Teachers Association, Phi Kappa Phi, Delta Kappa Gamma.

Activities: Former vice-president, California Mathematics Council; former county representative, California Mathematics Council; former member, NCTM Committee of Supplementary Publications; speaker at numerous mathematics meetings at both state and national levels; instructor for a week's course at the summer meeting of NEATM, 1955; chairman of local Mathematics Curriculum Committee, 1956–59.

Publications: Coauthor of textbooks, *Plane Geometry* and *High School Geometry;* article in *18th Yearbook* of the NCTM; articles in *Bulletin of the California Mathematics Council.*

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

THIRTY-NINTH ANNUAL MEETING

April 5–8, 1961
Conrad Hilton Hotel, Chicago, Illinois
Robert Sisler, Morton High School West, 2400 Home Avenue, Berwyn, Illinois

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

TWENTY-FIRST SUMMER MEETING

August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Canada

Other professional dates

The Greater Cleveland Council of Teachers of Mathematics
January 19, 1961
Western Reserve University
February 16, 1961
Roehm Junior High School, Berea, Ohio
Bessie Kisner, Strongsville High School, Strongsville, Ohio

Men's Mathematics Club of Chicago and Metropolitan Area
January 20, 1961
YMCA Hotel, South Wabash, Chicago
February 17, 1961
YMCA Hotel, South Wabash, Chicago
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

THE ELEMENTARY AND JUNIOR HIGH SCHOOL MATHEMATICS LIBRARY

by

Clarence Ethel Hardgrove

An annotated bibliography of selected recreational and informational materials for mathematics.

Classified by primary grades, intermediate grades, and junior high school.

Very useful to teachers and librarians.

Suggests readings to help your students grow in the ability to think with the ideas of mathematics.

32 pages 35¢ each

Postpaid if you send remittance with order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

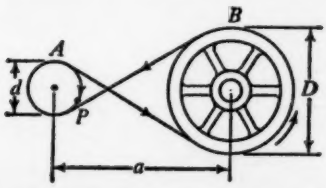
1201 Sixteenth Street, N.W. Washington 6, D.C.

1961 Editions on sale January 2nd!

FIRST COURSE IN ALGEBRA

SECOND COURSE IN ALGEBRA

MALLORY • MESERVE • SKEEN



MODERN IN APPROACH AND TERMINOLOGY . . . SOUND AND TEACHABLE

Write DEPT. N29 for complete information!

THE L.W. SINGER COMPANY

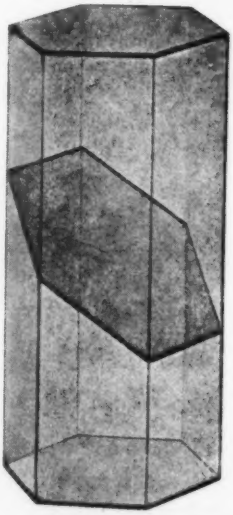
SYRACUSE 2, NEW YORK

NEW Welch

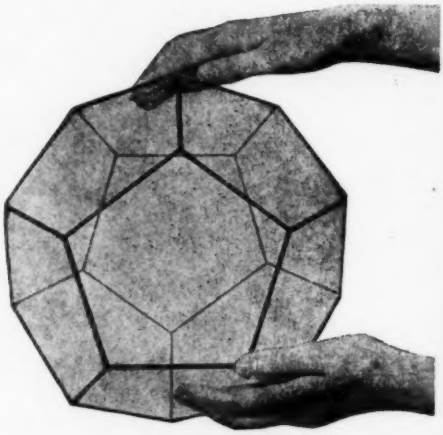
24 Transparent Mathematics Models Especially Suitable for Demonstration

NEW LARGE SIZE

Average One Foot High—Some as large as 15 inches



No. 7577 Hexagonal Prism
Each $9.50



No. 7592 Dodecahedron
Each $12.50

Permit the Class to see them in detail

Carefully made of Crystal Clear, heavy, transparent plastic with sections and some parts of brightly colored plastic.

Valuable teaching aids in plane and solid geometry, algebra, and other mathematics classes.

Degree of accuracy provides that measurements may be taken from them for numerical problems.

Write for circular describing all 24 models.
Serving the schools for 80 years.

THE WELCH SCIENTIFIC COMPANY

Established 1880
1515 Sedgwick St. Dept. X Chicago 10, Ill., U.S.A.
Manufacturers of Scientific Instruments and Laboratory Apparatus.